Old Dominion University Research Foundation

DEPARTMENT OF ELECTRICAL AND COMPUTER ENGINEERING
COLLEGE OF ENGINEERING
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NORFOLK, VIRGINIA 23508

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GUIDANCE AND CONTROL STRATEGIES FOR AEROSPACE VEHICLES

By

D. S. Naidu, Co-Principal Investigator

Principal Investigator: J. L. Hibey

Progress Report
For the period January 1, 1987 through June 30, 1987

Prepared for the
National Aeronautics and Space Administration
Langley Research Center
Hampton, Virginia 23665

Under
Research Grant NAG-1-737
Dr. Douglas B. Price, Technical Monitor
GCD-Spacecraft Controls Branch

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PROGRESS REPORT

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Dept. of Elect. & Computer Engg.,

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Summary of Research Work

The specification spectrum for the proposed Space Transportation System (STS) places heavy emphasis on the development of reusable avionics subsystems having special features such as vehicle evaluation and reduction of ground support for mission planning, contingency response and verification and validation. According to the recent report of the National Commission on Space, PIONEERING THE SPACE FRONTIER, the concept of aerobraking for orbit transfer has been recognized as one of the critical technologies and recommended for demonstration projects in building the necessary technology base for pioneering the space frontier.

As a first step in developing the necessary guidance and control strategies for aerospace vehicles, the dynamic equations of motion for both coplanar and noncoplanar Aeroassisted Orbit Transfer Vehicles (AOTV's) have been formulated in different ways using time, altitude, or energy as independent variable. The formulation with energy seems to be promising. Trajectory simulations have been obtained for these formulations, with particular emphasis on the effect of atmospheric density scale height on the performance of these vehicles. Simulations have shown that there is a considerable discrepancy between the plots with constant scale height and variable scale height. (see item (viii) in the enclosed list of publications).

A simplified method of matched asymptotic expansions has been developed where the common part in composite solution is generated as a polynomial in stretched variable instead of actually evaluating the same from outer solution. This methodology has been applied to the solution of the exact equations for three dimensional atmospheric entry problem. Here, it has been possible to obtain explicit relations between the constants of integration and the given initial conditions. This in in contrast to the earlier works where these relations led to a transcendental equation which can only be solved by resorting to numerical methods on a digital computer. (See item (x) in the enclosed list of publications).

Currently, a general optimization procedure using multiple shooting method for obtaining optimal guidance and control laws for orbital transfer vehicles, is being investigated, with a possibility of using the above mentioned simplified method of matched asymptotic expansions.

During the same period, several related research works have been carried out and are briefly mentioned below.

1. An overview of singular perturbations and time scales (SPaTS) in discrete control systems has been conducted focusing in three directions of modeling, analysis and control. The resulting tutorial-cum-survey paper has been accepted for presentation at an invited session, at IEEE Conference on Decision and Control, Los Angeles, CA, December 9-11, 1987. This session is being organized and chaired by Dr. D. S. Naidu, the co-principal investigator. A draft copy of the paper is enclosed. (See item (v) in the list of publications).

2. An important work in the same period is the final preparation of the forthcoming book entitled, "SINGULAR PERTURBATION METHODOLOGY IN CONTROL SYSTEMS, authored by Dr. D. S. Naidu, the co-principal investigator, and being published under IEE Control Engineering Series, by Peter Peregrinus Limited, Stevenage Herts, England. This book is scheduled to appear in September 1987.(See item (i) in the enclosed list of publications).

3. As an outgrowth of earlier work on singular perturbations and time scales in discrete control systems, it has been found that to a zeroth order approximation, these two approaches yield identical results. (See item (vi) in the enclosed list of publications).

4. Other works are concerned with the items (ii), (iii), (iv), and (vii) in the list of publications.

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List of Publications

(i) D. S. Naidu, "Singular Perturbation Methodology in Control Systems", IEE Control Engineering Series, Peter Peregrinus Ltd., Stevenage Herts, England, 1987. (in press)

(ii) D. S. Naidu and D. B. Price, "Time scale synthesis of a closed-loop discrete optimal control system", Journal of Guidance, Control, and Dynamics, 10, 1987. (in press)

(iii) L. W. Taylor, Jr., and D. S. Naidu, "Experience in distributed parameter modeling of the spacecraft control laboratory experiment (SCOLE) structure", AIAA Dynamics Specialists Conference, Monterey, CA, April 1987.

(iv) D. S. Naidu and M. S. K. Rayalu, "Singular perturbation method for initial value problems in two-parameter discrete control systems", Int. J. Systems Science, 18, 1987 (in press).

*(v) D. S. Naidu, D. B. Price and J. L. Hibey, "Singular perturbations and time scales in discrete control systems-an overview", Accepted for presentation at the Invited Session, IEEE Conference on Decision and Control, Los Angles, CA, Dec. 9-11, 1987

*(vi) D. S. Naidu and D. B. Price, "On singular perturbation and time scale approaches in discrete control systems", communicated to Journal of Guidance, Control and Dynamics, June 1987.

(vii) D. S. Naidu and D. B. Price, "Singular perturbations and time scales in digital flight control systems", NASA Technical Publication, Langley Research Center, Hampton (in preparation).

*(viii) D. S. Naidu and D. B. Price, "Impact of atmospheric scale height on the performance of aeroassisted orbiter transfer vehicles", Spacecraft Controls Branch, NASA Langley Research Center, Hampton, May 1987.

(ix) D. S. Naidu and S. Sen, "A time-optimal control algorithm for two-time scale discrete system", communicated to Int. J. Control, (1987).

*(x) D. S. Naidu and D. B. Price, "On the method of matched asymptotic expansions", Accepted for presentation at the SIAM Annual Meeting and 35th Anniversary, Denver, CO, October 12-15, 1987.

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ON SINGULAR PERTURBATION AND TIME SCALE APPROACHES IN DISCRETE CONTROL SYSTEMS

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For Submission as a Technical Note to the
AIAA Journal of Guidance, Control, and Dynamics

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ON SINGULAR PERTURBATION AND TIME SCALE APPROACHES IN DISCRETE CONTROL SYSTEMS

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INTRODUCTION

The theory of singular perturbations and time scales (SPATS) has been a powerful analytical tool in the analysis and synthesis of continuous and discrete control systems.[1,2] In this technical note, we first consider a singularly perturbed discrete control system. Using a singular perturbation approach, outer and correction subsystems are obtained. Next, by the application of time scale approach via block diagonalization transformations, the original system is decoupled into slow and fast subsystems. It will be shown that to a zeroth order approximation, the singular perturbation and time scale approaches yield equivalent results. Roughly speaking, the zeroth-order approximation is sometimes called the first approximation. This result is similar to a corresponding result in continuous control systems.[3]

SINGULAR PERTURBATION APPROACH

Consider a general form for linear, shift-invariant, singularly perturbed discrete systems as[2]

$$x(k+1) = A_{11}x(k) + h^{1-j}A_{12}z(k) + B_1u(k) \tag{1a}$$

$$h^{2i}z(k+1) = h^jA_{21}x(k) + hA_{22}z(k) + h^jB_2u(k) \tag{1b}$$

$$0 \le i \le 1; \quad 0 \le j \le 1$$

where, $x(k)$ and $z(k)$ are "slow" and "fast" state vectors of n and m dimensions respectively, $u(k)$ is an r-dimensional control vector, h is a singular perturbation parameter, and A's and B's are matrices of appropriate dimensionality. We formulate initial value problems with $x(k=0) = x(0)$ and $z(k=0) = z(0)$ and note that similar results can be obtained for boundary value problems also.

The three limiting cases of Eq. (1) result in

(1) the C-model ($i=0$; $j=0$),

$$x(k+1) = A_{11}x(k) + hA_{12}z(k) + B_1u(k) \tag{2a}$$

$$z(k+1) = A_{21}x(k) + hA_{22}z(k) + B_2u(k) \tag{2b}$$

where the small parameter h appears in the column of the system matrix,

(2) the R-model ($i=0$; $j=1$),

$$x(k+1) = A_{11}x(k) + A_{12}z(k) + B_1u(k) \tag{3a}$$

$$z(k+1) = hA_{21}x(k) + hA_{22}z(k) + hB_2u(k) \tag{3b}$$

where the small parameter h appears in the row of the system matrix, and

(3) the D-model ($i=1$; $j=1$),

$$x(k+1) = A_{11}x(k) + A_{12}z(k) + B_1u(k) \tag{4a}$$

$$hz(k+1) = A_{21}x(k) + A_{22}z(k) + B_2u(k) \tag{4b}$$

where the small parameter h is positioned in an identical fashion to that of the continuous systems described by <u>differential</u> equations. In this note, we consider only the C-model of Eq. (2), but the result can be extended to the other two models of Eqs. (3) and (4) as well. The outer (degenerate) subsystem, obtained by zeroth-order approximation (i.e., by making $h=0$) of Eq. (2), is

$$x^{(o)}(k+1) = A_{11}x^{(o)}(k) + B_1u^{(o)}(k) \tag{5a}$$

$$z^{(o)}(k+1) = A_{21}x^{(o)}(k) + B_2u^{(o)}(k) \tag{5b}$$

$$x^{(o)}(k=0) = x(o); \quad z^{(o)}(k=0) \neq z(0) \tag{5c}$$

Here, we note that in the process of degeneration, x(k) has retained its initial condition x(0), whereas z(k) has lost its initial condition z(0). In order to recover this lost initial condition, a correction subsystem is used.[2] The transformations between the original and correction variables are

$$x_c(k) = x(k)/h^{k+1}; \quad z_c(k) = z(k)/h^k \tag{6a}$$

$$u_c(k) = u(k)/h^{k+1} \tag{6b}$$

Using Eq. (6) in Eq. (2), the transformed system becomes,

$$h x_c(k+1) = A_{11} x_c(k) + A_{12} z_c(k) + B_1 u_c(k) \tag{7a}$$

$$z_c(k+1) = A_{21} x_c(k) + A_{22} z_c(k) + B_2 u_c(k) \tag{7b}$$

The zeroth-order approximation (h=0) of Eq. (7) becomes,

$$0 = A_{11} x_c^{(o)}(k) + A_{12} z_c^{(o)}(k) + B_1 u_c^{(o)}(k) \tag{8a}$$

$$z_c^{(o)}(k+1) = A_{21} x_c^{(o)}(k) + A_{22} z_c^{(o)}(k) + B_2 u^{(o)}(k) \tag{8b}$$

Rewriting Eq. (8), we get,

$$x_c^{(o)}(k) = -A_{11}^{-1} [A_{12} z_c^{(o)}(k) + B_1 u_c^{(o)}(k)] \tag{9a}$$

$$z_c^{(o)}(k+1) = A_{co} z_c^{(o)}(k) + B_{co} u_c^{(o)}(k) \tag{9b}$$

where,

$$A_{co} = A_{22} - A_{21} A_{11}^{-1} A_{12}$$

$$B_{co} = B_2 - A_{21} A_{11}^{-1} B_1$$

The total solution consists of an outer solution and a correction solution as[2]

$$x(k) = [x^{(o)}(k) + h x^{(1)}(k) + \dots] + h^{k+1} [x_c^{(o)}(k) + x_c^{(1)}(k) + \dots] \tag{10a}$$

$$z(k) = [z^{(o)}(k) + h z^{(1)}(k) + \dots] + h^k [z_c^{(o)}(k) + z_c^{(1)}(k) + \dots] \tag{10b}$$

For the present, to simplify the analysis, we omit u(k) and its associated functions. Then for zeroth-order approximation, the total solution is given by[2]

$$x(k) = x^{(o)}(k) \tag{11a}$$

$$z(k) = z^{(o)}(k) + h^k z_c^{(o)}(k) \tag{11b}$$

$$= z^{(o)}(k) + z_r^{(o)}(k) \tag{11c}$$

where, $z_r^{(o)}(k) = h^k z_c^{(o)}(k)$. From Eq. (5c), we note that only z(k) has lost its initial condition. Hence Eq. (11) gives

$$z_c^{(o)}(k{=}0) = z(0) - z^{(o)}(0) \tag{12}$$

Our current interest is only zeroth-order approximations. Thus, from Eqs. (5) and (9), we get

$$x^{(o)}(k+1) = A_{11}x^{(o)}(k) \tag{13a}$$

$$z^{(o)}(k+1) = A_{21}A_{11}^{-1}x^{(o)}(k+1) \tag{13b}$$

or $$z^{(o)}(k) = A_{21}A_{11}^{-1}x^{(o)}(k) \tag{13c}$$

and the correction functions as,

$$z_c^{(o)}(k+1) = A_{co}z_c^{(o)}(k) \tag{14a}$$

or $$z_r^{(o)}(k+1) = hA_{co}z_r^{(o)}(k) \tag{14b}$$

where, $z_r^{(o)}(k{=}0) = z_c^{(o)}(0)$

$$= z(0) - z^{(o)}(0)$$

$$= z(0) - A_{21}A_{11}^{-1}x(0)$$

TIME SCALE APPROACH

Let us consider again the singularly perturbed system of Eq. (2). We now use the time scale approach and obtain slow and fast subsystems to a zeroth-order approximation.

For decoupling the original system of Eq. (2) into slow and fast subsystems, the block diagonalization transformations relating the decoupled variables in terms of the original variables are[4]

$$x_s(k) = (I_s + hED)x(k) + hEz(k) \tag{15a}$$

$$z_f(k) = Dx(k) + I_f z(k) \tag{15b}$$

and transformations relating the original variables and the decoupled variables are

$$x(k) = x_s(k) - hEz_f(k) \tag{16a}$$

$$z(k) = -Dx_s(k) + (I_f + hDE)z_f(k) \tag{16b}$$

Where $I_s(nxn)$ and $I_f(mxm)$ are unity matrices and $D(mxn)$ and $E(nxm)$ satisfy Riccati-type algebraic equations,

$$hA_{22}D - DA_{11} + hDA_{12}D - A_{21} = 0 \tag{17a}$$

$$hE(A_{22} + DA_{12}) - (A_{11} - hA_{12}D)E + A_{12} = 0 \tag{17b}$$

whose iterative solutions start with initial values of $D_i = -A_{21}A_{11}^{-1}$ and $E_i = A_{11}^{-1}A_{12}$. By using transformations given by Eq. (15) in Eq. (2), we get the decoupled slow and fast subsystems as,

$$x_s(k+1) = A_s x_s(k) + B_s u(k) \tag{18a}$$

$$z_f(k+1) = hA_f z_f(k) + B_f u(k) \tag{18b}$$

where, $A_s = A_{11} - hA_{12}D$; $A_f = A_{22} + DA_{12}$

$B_s = (I_s + hED)B_1 + hEB_2$

$B_f = DB_1 + B_2$

For zeroth-order approximation,[3] we get,

$$D_o = -A_{21}A_{11}^{-1}; \quad E_o = A_{11}^{-1}A_{12} \tag{19a}$$

$$A_{so} = A_{11}; \quad A_{fo} = A_{22} - A_{21}A_{11}^{-1}A_{12} \tag{19b}$$

$$B_{so} = B_1; \quad B_{fo} = B_2 - A_{21}A_{11}^{-1}B_1 \tag{19c}$$

Using Eq. (19) in Eq. (16) and (18), we get (omitting input for simplicity),

$$x(k) = x_s^{(o)}(k) \tag{20a}$$

$$z(k) = A_{21}A_{11}^{-1}x_s^{(o)}(k) + z_f^{(o)}(k) \tag{20b}$$

where $x_s^{(o)}(k)$ and $z_f^{(o)}(k)$ satisfy

$$x_s^{(o)}(k+1) = A_{11}x_s^{(o)}(k) \tag{21a}$$

$$z_f^{(o)}(k+1) = hA_{fo}z_f^{(o)}(k) \tag{21b}$$

Similarly, using Eq. (19) in Eq. (15), we obtain,

$$x_s^{(o)}(k=0) = x(0); \tag{22a}$$

$$z_f^{(o)}(k=0) = z(0) - A_{21}A_{11}^{-1}x(0) \tag{22b}$$

Comparing the subsystems of Eqs. (13) and (14) and the solution of Eq. (11) obtained by using the singular perturbation approach with the corresponding subsystems of Eq. (21) and the solution of Eq. (20), we find that they satisfy the same equations with the same initial conditions. Hence,

$$x^{(o)}(k) = x_s^{(o)}(k); \quad z^{(o)}(k) = A_{21}A_{11}^{-1}x_s^{(o)}(k) \tag{23a}$$

$$z_r^{(o)}(k) = z_f^{(o)}(k) \qquad A_{co} = A_{fo} \tag{23b}$$

Thus, we have shown that for a zeroth-order approximation, both singular perturbation and time scale approaches give identical results. Similar results can be established for other types of discrete systems characterized by Eqs. (3) and (4).

CONCLUSION

In this note, we have demonstrated for a zeroth-order approximation the equivalence of the subsystems obtained by the singular perturbation and time scale approaches. This result is akin to that in the singularly perturbed continuous systems. It has been seen that such an equivalence does exist for a first-order approximation also, the details of which are omitted due to the lengthy and cumbersome nature of the derivations.

REFERENCES

[1]Saksena, V. R., O'Reilly, J., and Kokotovic, P. V., "Singular Perturbations and Time-Scale Methods in Control Theory: Survey," Automatica, Vol. 220, pp. 273-293, 1984.

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[3]Mahmoud, M. S., and Singh, M. G., Large Scale Systems Modeling, Pergamon Press, Oxford, 1981.

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SINGULAR PERTURBATIONS AND TIME SCALES (SPTS) IN DISCRETE CONTROL SYSTEMS-AN OVERVIEW

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Abstract:

This paper overviews recent developments in the theory of singular perturbations and time scales (SPTS) in the discrete control systems. The focus is in three directions of modeling, analysis and control. First, we review sources of discrete models and the effect of discretizing interval on the modeling. The analysis of two-time scale systems brings out typical charactetistic features of SPTS. In controlling the two-time scale systems, we address the important issue of multirate sampling. The bibliography contains over 100 titles.

1. INTRODUCTION

The dynamics of many control systems is described by high order differential equations. However, the behaviour in governed by a few dominant parameters, a relatively minor role being played by the remaining parameters such as small time constants, masses, moments of inertia, inductances, and capacitances. The presence of these "parasitic" parameters is often the source for the increased order and the "stiffness" of the system. The "curse" of the dimensionality coupled with stiffness poses formidable computational complexities for the analysis and control of such large systems. The methodology of singular perturbations and time scales (SPTS) is a "gift" to control engineers. As such it is very desirable to formulate many control problems to fit into the framework of the mathematical theory of SPTS which has a rich literature (Van Dyke 64, Wasow 65, Cole 68, Butuzov et. el., 70, Eckhaus 73, 79, Nayfeh 73, 81, Vasileva and Butuzov 73, 78, Nayfeh and Mook 79, Eckhaus and de Jager 82, Chang and Howes 84, Smith 85). The theory of SPTS in continuous control systems has attained a reasonable level of maturity and is well documeted (Kokotovic and Perkins 72, O'Malley 74, Genesio and Milanese 76, Kokotovic et. el., 76, 86, Ardema 83, Kokotovic 84, 85, Saksena et. el., 84, Naidu 87).

The methodology of SPTS has an impressive record of applications in a wide spectrum of fields such as circuits (Sastry and Desoer 81), networks (Sannuti 81), electrostatics (Abraham-Shrauner 74), electromagnetics (Seshadri 76), electrical mechines (Zaid et. el., 82), power systems (Chow 82), semiconductors (Markowich and Ringhoffer 84), fluid mechanics (Van Dyke 64), structural mechanics (Flaherty and O'Malley 82), soil mechanics (Dicker and Babu 74), flight mechanics (Ardema 77), celestial mechanics (Verhulst 75), geophysics (Carrier 70), chemistry (Cohen 74), thermodynamics (Cooper 75), nuclear reactor dynamics (Reddy and Sannuti 75), accoustics (Einaudi 69), oceanography (Ruijter 79), biology (Carpenter 77), biochemistry (Heineken et. el., 67), elology (Naidu and

Rajagopalan 79), lasers (Eckhaus et. el., 85), and robotics (Chernousko and Shamaev 83).

Discrete systems are very much prevelent in science and engineering. There are three sources of discrete models described by difference equations containing several parameters (Dorato and Levis 71). The first source is digital simulation, where ordinary differntial equations are approximated by the corresponding difference equations (Hildebrand 68, Abrahamsson et. el., 74, Hemker and Miller 79, Miranker 80). The study of sampled-data control systems and computer-based adaptive control systems leads in a natural way to another source of discrete-time models (Kuo 80). Finally, many economic, biological and sociological systems are represented by discrete models (Cadzow 73). In spite of the fact that the digital control of systems with widely separated eigenvalues was first considered by stineman (65), the field of singular perturbations and time scales in difference equations and its applications to discrete control systems is of recent orgin only (Comstock and Hsiao 76, Locatelli and Schiavoni 76, Hoppensteadt and Miranker 77, Naidu 77, Vasileva and Faminskaya 77, Javid 79, Reinhardt 79, Phillips 80, Rajagopalan and Naidu 80, Atluri and Kao 81, Blankenship 81).

This paper overviews these recent developments in the theory of SPTS in difference equations and discrete control systems. The focus is on three directions of modelling, analysis and control.

2. MODELING IN SPTS SYSTEMS:

2.1. Source I: Pure Difference Equations:

Consider a general linear, shift-invariant difference equation with small parameters occuring at the right end, left end or both ends. Then the state variable model becomes (Syrcos and Sannuti 83, Naidu and Rao 81, 82),

$$x_1(k+1) = A_{11}x_1(k) + h^{1-j}A_{12}x_2(k) + B_1u(k) \qquad (1a)$$

$$h^{2i}x_2(k+1) = h^jA_{21}x_1(k) + hA_{22}x_2(k) + B_2u(k) \qquad (1b)$$

$$0 < i < 1; \quad 0 < j < 1$$

where, $x_1(k)$ and $x_2(k)$ are "slow" and "fast" state vectors of n_1 and n_2 dimensions respectively, $u(k)$ is an r dimensional control vector, h is singular perturbation parameter and A's and B's are matrices of appropriate dimensionality.

The three limiting cases of (1) result in

(i) the C-model ($i=0$; $j=0$),

$$x_1(k+1) = A_{11}x_1(k) + hA_{12}x_2(k) + B_1u(k) \tag{2a}$$

$$x_2(k+1) = A_{21}x_1(k) + hA_{22}x_2(k) + B_2u(k) \tag{2b}$$

where the small parameter h appears in the column of the system matrix,

(ii) the R-model ($i=0$; $j=1$),

$$x_1(k+1) = A_{11}x_1(k) + A_{12}x_2(k) + B_1u(k) \tag{3a}$$

$$x_2(k+1) = hA_{21}x_1(k) + hA_{22}x_2(k) + B_2u(k) \tag{3b}$$

where the small parameter h appears in the row of the system matrix, and

(iii) the D-model ($i=1$; $j=1$),

$$x_1(k+1) = A_{11}x_1(k) + A_{12}x_2(k) + B_1u(k) \tag{4a}$$

$$hx_2(k+1) = A_{21}x_1(k) + A_{22}x_2(k) + B_2u(k) \tag{4b}$$

where the small parameter h is positioned in an identical fashion to that of the continuous systems described by differential equations. Note: The replacement of $x_2(k)$ by $hx_1(k)$ in model (3) will result in model (2).

2.2 Source II: Discrete Modelling of Continous Systems:

Here either numerical solution or sampling of singularly perturbed continuous systems will result in discrete models. Consider the singularly perturbed continuous system as

$$x_1(t) = A_{11}x_1(t) + A_{12}x_2(t) + B_1u(t) \tag{5a}$$

$$hx_2(t) = A_{21}x_1(t) + A_{22}x_2(t) + B_2u(t) \tag{5b}$$

Applying the block-diagonalization transformations (Phillips 80), the original variables $x_1(t)$ and $x_2(t)$ are expressed in terms of the decoupled variables $x_s(t)$ and $x_f(t)$ as

$$x_1(t) = I_sx_s(t) - hMx_f(t) \tag{6a}$$

$$x_2(t) = -Lx_s(t) + (I_f+hLM)x_f(t) \tag{6b}$$

and the decoupled variables $x_s(t)$ and $x_f(t)$ are obtained in terms of the original variables $x_1(t)$ and $x_2(t)$ as

$$x_s(t) = (I_s+hML)x_1(t) + hMx_2(t) \tag{7a}$$

$$x_f(t) = Lx_1(t) + I_fx_2(t) \tag{7b}$$

where L and M satisfy

$$A_{21} + hLA_{11} - A_{22}L - hLA_{22}L = 0 \tag{8a}$$

$$A_{12} - h(A_{11}-A_{12}L)M + M(A_{22}+hLA_{12}) = 0 \tag{8b}$$

Using (6) and (7) to the continuous system (5) with a sample-hold device, we get a discrete model, which depends critically on the sampling interval T (Kando 86). If we choose the fast sampling T_f = h or the slow sampling T_s = $1[1/h]T_f$ (where [1/h] is the largest integer < 1/h), we get fast or slow sampling model. In a particular case, when T_f = h, we get the <u>fast sampling model</u> as

$$x_1(n+1) = (I_s+hD_{11})x_1(n) + hD_{12}x_2(n) + hD_1u(n) \tag{9a}$$

$$x_2(n+1) = D_{21}x_1(n) + D_{22}x_2(n) + D_2u(n) \tag{9b}$$

where n denotes the fast sampling instant. Similarly, if T_s = 1, we obtain the <u>slow sampling model</u> as

$$x_1(k+1) = E_{11}x_1(k) + hE_{12}x_2(k) + E_1u(k) \tag{10a}$$

$$x_2(k+1) = E_{21}x_1(k) + hE_{22}x_2(k) + E_2u(k) \tag{10b}$$

where k represents the slow sampling point, and n = k[1/h]. Also, the D's and E's are related with A's, B's, L and M.

<u>Remarks:</u>

(i) the fast sampling model (9) can be viewed as the discrete-time analog (either by exact calculation using exponential matrix or by using Euler approximation) of the continuous system

$$\dot{x}_1(t') = hA_{11}x_1(t') + hA_{12}x_2(t') + hB_1u(t') \tag{11a}$$

$$x_2(t') = A_{21}x_1(t') + A_{22}x_2(t') + B_2u(t') \tag{11b}$$

which itself is obtained from the continuous system (5) using stretching transformation t' = t/h. It is usually said that the singularly continuous perturbed systems (5) and (10) are the slow time scale (t) and the fast time scale (t') versions respectively.

(ii) the slow sampling model (10) is the same as the state space model (2) obtained from the singularly perturbed difference equations (Comstock and Hsio 76, Naidu and Rao 81). Thus by discretizing the singularly perturbed continuous system (5) with slow and fast sampling rates, we get two different discrete-time models.

<u>Time Scale Property</u>

The slow sampling model (10) possesses the two-time scale property, if the largest eigenvalue of E_f is much smaller than the smallest eigenvalue of E_s, that is (Phillips 80),

$$\max | (E_f) | << \min | (E_s) | \quad (12a)$$

$$\text{or} \quad 1 > |p_1| > \ldots |p_{n1}| >> |p_{n1+1}| > \ldots |p_{n1+n2}| \quad (12b)$$

where the approximate values for E_s and E_f are

$$E_s = E_{11} + O(h) \quad (12a)$$

$$E_f = h(E_{22} - E_{11}^{-1}E_{12}) + O(h^2) \quad (12b)$$

Simialarly, we can obtain the condition for the fast sampling model (9) to exhibit the two-time scale property (Blankenship 81, and Kando 86).

3. ANALYSIS IN SPTS SYSTEMS:

In this section, we analyze the systems using singular perturbation and time-scale approaches, and show that the two approaches give identical results.

We first consider a singularly perturbed discrete control system. Using singular perturbation approach, outer and correction subsystems are obtained. Next, by the application of time scale approach via block diagonalization transformations, the original system is decoupled into slow and fast subsystems. To a zeroth order approximation, the singular perturbation and time scale approaches yield equivalent results. This result is similar to a corresponding result in continuous control systems(Mahmoud and Singh 81).

3.1 Slow Sampling Model: Intial Value Problems (IVP)

3.1.1 Singular Perturbation Approach

Consider the singularly perturbed discrete system (2). We formulate initial value problem and note that similar result can be obtained for boundary value problems also.

The outer (degenerate) subsystem, obtained by zeroth order approximation (i.e., by making h=0) of (2), is

$$x_1^{(o)}(k+1) = A_{11}x_1^{(o)}(k) + B_1u^{(o)}(k) \quad (14a)$$

$$x_2^{(o)}(k+1) = A_{21}x_1^{(o)}(k) + B_2u^{(o)}(k) \quad (14b)$$

$$x_1^{(o)}(k=0) = x(0); \quad x_2^{(o)}(k=0) = x_2(0) \quad (14c)$$

Here, we note that in the process of degeneration, $x_1(k)$ has retained its initial condition $x_1(0)$, whereas $x_2(k)$ has lost its initial condition $x_2(0)$. The boundary layer is said to exist at $k=0$. In order to recover this lost initial condition, a correction subsystem is used (Naidu and Rao 85). The transformations between the original and correction variables are (assuming no inputs for simplicity),

$$x_{1c}(k) = x_1(k)/h^{k+1}; \quad x_{2c}(k) = x_2(k)/h^k \tag{15}$$

Using (15) in (2), the transformed system becomes,

$$hx_{1c}(k+1) = A_{11}x_{1c}(k) + A_{12}x_{2c}(k) \tag{16a}$$

$$x_{2c}(k+1) = A_{21}x_{1c}(k) + A_{22}x_{2c}(k) \tag{16b}$$

The zeroth order approximation (h=0) of (15) becomes,

$$0 = A_{11}x_{1c}^{(0)}(k) + A_{12}x_{2c}^{(0)}(k) \tag{17a}$$

$$x_{2c}^{(0)}(k+1) = A_{21}x_{1c}^{(0)}(k) + A_{22}x_{2c}^{(0)}(k) \tag{17b}$$

Rewriting (17), we get,

$$x_{1c}^{(0)}(k) = -A_{11}^{-1}[A_{12}x_{2c}^{(0)}(k)] \tag{18a}$$

$$x_{2c}^{(0)}(k+1) = A_{c0}x_{2c}^{(0)}(k) \tag{18b}$$

where, $\quad A_{c0} = A_{22} - A_{21}A_{11}^{-1}A_{12};$

The total solution consists of outer solution and correction solution as

$$x_1(k) = [x_1^{(0)}(k) + hx_1^{(1)}(k) + \ldots]$$
$$+h^{k+1}[x_{1c}^{(0)}(k) + x_{1c}^{(1)}(k) + \ldots] \tag{19a}$$

$$x_2(k) = x_2^{(0)}(k) + hx_2^{(1)}(k) + \ldots]$$
$$+h^k[x_{2c}^{(0)}(k) + x_{2c}^{(1)}(k) + \ldots] \tag{19b}$$

For zeroth order approximation, the total solution is given by

$$x_1(k) = x_1^{(0)}(k) \tag{20a}$$

$$x_2(k) = x_2^{(0)}(k) + h^kx_{2c}^{(0)}(k) \tag{20b}$$

$$= x_2^{(0)}(k) + x_{2r}^{(0)}(k) \tag{20c}$$

where, $x_{2r}^{(0)}(k) = h^kx_{2c}^{(0)}(k)$. From (14c), we note that only $z(k)$ has lost its initial condition. Hence (20) gives $x_{2c}^{(0)}(k=0) = x_2(0) - x_2^{(0)}(0)$.

Our current interest is only zeroth order approximations. Thus, from (14) and (18), we get

$$x_1^{(0)}(k+1) = A_{11}x_1^{(0)}(k) \tag{21a}$$

$$x_2^{(0)}(k+1) = A_{21}A_{11}^{-1}x_1^{(0)}(k+1) \tag{21b}$$

$$\text{or} \quad x_2^{(0)}(k) = A_{21}A_{11}^{-1}x_1^{(0)}(k) \tag{21c}$$

and the correction functions as,

$$x_{2c}^{(0)}(k+1) = A_{c0}x_{2c}^{(0)}(k) \tag{22a}$$

$$\text{or} \quad x_{2r}^{(0)}(k+1) = hA_{c0}x_{2r}^{(0)}(k) \tag{22b}$$

where, $x_{2r}^{(0)}(k=0) = x_{2c}^{(0)}(0)$

$$= x_2(0) - x_2^{(0)}(0)$$

$$= x_2(0) - A_{21}A_{11}^{-1}x_1(0)$$

3.1.2 Time Scale Approach:

Let us consider again the singularly perturbed system (2). We now use the time scale approach and obtain slow and fast subsystems to a zeroth order approximation.

For decoupling the original system (2) into slow and fast subsystems, the block diagonalization transformations relating the decoupled variables in terms of the original variables are (Phillips 80, Kando and Iwazumi 83),

$$x_s(k) = (I_s + hML)x_1(k) + hMx_2(k) \tag{23a}$$

$$x_f(k) = Lx_1(k) + I_f x_2(k) \tag{23b}$$

and transformations relating the original variables and the decoupled variables are

$$x_1(k) = I_s x_s(k) - hMx_f(k) \tag{24a}$$

$$x_2(k) = -Lx_s(k) + (I_f + hLM)x_f(k) \tag{24b}$$

where $L(n_1 \times n_2)$ and $M(n_1 \times n_2)$ satisfy Riccati type algebraic equations,

$$hA_{22}L - LA_{11} + hLA_{12}L - A_{21} = 0 \tag{25a}$$

$$hM(A_{22} + LA_{12}) - (A_{11} - hA_{12}L)M + hA_{12} = 0 \tag{25b}$$

whose iterative solutions start with initial values of $L_1 = -A_{21}A_{11}^{-1}$ and $M_1 = hA_{11}^{-1}A_{12}$. By using transformations (23) in (2), we get the decoupled slow and fast subsystems as,

$$x_s(k+1) = A_s x_s(k) + B_s u(k) \tag{26a}$$

$$x_f(k+1) = hA_f x_f(k) + B_f u(k) \tag{26b}$$

where, $A_s = A_{11} - hA_{12}L;$

$$B_s = (I_s + hML)B_1 + hMB_2; \quad B_f = LB_1 + B_2$$

For zeroth order approximation, we get,

$$Lo = -A_{21}A_{11}^{-1}; \quad Mo = 0; \tag{27a}$$

$$A_so = A_{11}; \quad A_{fo} = A_{22} - A_{21}A_{11}^{-1}A_{12}; \tag{27b}$$

$$B_{so} = B_1; \quad B_{fo} = B_2 - A_{21}A_{11}^{-1}B_1 \tag{27c}$$

Using (27) in (24) and (26), we get (omitting input for simplicity),

$$x_1(k) = x_{so}(k) \tag{28a}$$

$$x_2(k) = A_{21}A_{11}^{-1}x_{so}(k) + x_{fo}(k) \tag{28b}$$

where $x_{so}(k)$ and $x_{fo}(k)$ satisfy

$$x_{so}(k+1) = A_{11}x_{so}(k) \tag{29a}$$

$$x_{fo}(k+1) = hA_{fo}x_{fo}(k) \tag{29b}$$

Similarly, using (27) in (23), we obtain,

$$x_{so}(k=0) = x_1(0); \tag{30a}$$

$$x_{fo}(k=0) = x_2(0) - A_{21}A_{11}^{-1}x_1(0) \tag{30b}$$

Comparing the subsystems (21) and (22) and the solution (20) obtained by using the singular perturbation approach with the corresponding subsystems (29) and the solution (28), we find that they satisfy the same equations with the same initial conditions. Hence,

$$x_1^{(o)}(k) = x_{so}(k); \quad x_2^{(o)}(k) = -A_{21}A_{11}^{-1}x_{so}(k) \tag{31a}$$

$$x_{2r}^{(o)}(k) = x_{fo}(k) \qquad A_{co} = A_{fo} \tag{31b}$$

Thus, we have shown that for a zeroth order approximation, both singular perturbation and time scale approaches give identical results.

Thus, we have found that for a zeroth order approximation the equivalence of the subsystems obtained by the singular perturbation and time scale approaches. This result is akin to that in the singularly perturbed continuous systems. It has been seen that such an equivalence does exist for first and high order approximations also (Kando 86).

In the slow sampling model, the solution can be expressed as a combination of discrete-time slow and fast subsystems. Here, the two-time scale property of the discrete-time itself, and the lower sampling rate are assumed. However, it is noted that the fast part is treated as dead-beat. As a result, the slow sampling model (9) is obtained from the continuous system (5), there is bound to be performance degradation between the two systems over the intial interval only.

3.2 Boundary Value Problems (BVP):

The analysis of BVP is similar to that of IVP, with few differences which are described below. For both C- or R-models, if the boundary conditions are $x_1(N)$ and $x_2(0)$, then the boundary layer still occur at k=0, and the total series solution still remains the same as (18) (Naidu and Rao 81, 82, 85a,b, Rao and Naidu 81). However, the auxiliary condtions are

$$x^{(0)}(k=N) = x_1(N); \quad x_{2c}(k=0) = x_2(0) - x_2^{(0)}(0) \qquad (32)$$

For the D-model, if the boundary condtions are $x_1(0)$ and $x_2(N)$, the total series solution is given by

$$x_1(k) = [x_1^{(0)}(k) + hx_1^{(1)}(k) + \ldots]$$

$$+h^{N-k+1}[x_{1c}^{(0)}(k) + x_{1c}^{(1)}(k) + \ldots] \qquad (33a)$$

$$x_2(k) = x_2^{(0)}(k) + hx_2^{(1)}(k) + \ldots]$$

$$+h^{N-k}[x_{2c}^{(0)}(k) + x_{2c}^{(1)}(k) + \ldots] \qquad (33b)$$

and the boundary layer is said to exist at the final point k = N.

3.3 Fast Sampling Model:

Consider the fast-sampling model (10), which is more exact model than the Euler approximation model of Blankeship (81) and Rajagopalan and Naidu (81).

The eigenvalues of the slow and fast parts of the fast-sampling model (10) are given by

$$p(D_s) = p\{I+h(D_{11}-D_{12}L)\} \qquad (34a)$$

$$p(D_f) = p\{D_{22}+hLD_{12}\} \tag{34b}$$

where L is the dichotomic solution of a nonlinear algebraic Riccati equation (NARE),

$$D_{12} - D_{22}L + L(I+hD_{11}) - hLD_{12}L = 0 \tag{35}$$

It is noted that even if the continuous-time system (5) possesses the two-time scale property, i.e.,

$$\max |p(A_s)| << \min |p(A_f)| \tag{36a}$$

the fast sampling model (10) does not necessarily satisfy its two-time scale property, i.e.,

$$\min |p(D_s)| >> \max |p(D_f)| \tag{36b}$$

This is in contrast to the slow-sampling model (2) or (10), which preserves its two-time scale property in the discretization process.

Using a boundary layer method, the solutions of (10) are expressed as

$$x_1(n,h) = X_1(t,h) + hx_{1c}(n,h), \quad t = hn, \tag{37a}$$

$$x_2(n,h) = X_2(t,h) + x_{2c}(n,h), \tag{37b}$$

$$u(n,h) + U(t,h) + u_c(n,h) \tag{37c}$$

where $X_1(t,h)$, $X_2(t,h)$ and $U(t,h)$ correspond to reduced system of the continuous system (5).

Thus the solution (37) of (10) can be expressed as a hybrid combination of the continuous slow part which dominates the system behaviour over whole interval, and the discrete-time fast part which dominates over the intial time only. Thus, the analysis and design are performed essentially in the continuous-time domain.

3.4 Steady State Analysis

An alternative approach to deriving the slow and fast subsystems is based on quasi-steady state concepts (Badreddin 82, Mahmoud 82, Tran and Sawan 83a,b, 84a,b,c). For a stable linear discrete system having the time-scale property, the fast modes corresponding to the eigenvalues centered around the origin, are important only during the first few discrete instants (transient period). After that period, they are negligible and the slow modes dominate the behaviour of the discrete systems.

Neglecting the effects of fast modes is expressed formally by letting $x_2(k+1) = x_2(k)$. Then, we get,

$$\underline{x}_1(k+1) = \underline{A}_{11}\underline{x}_1(k) + \underline{A}_{12}\underline{x}_2(k) + B_1\underline{u}(k) \qquad (38a)$$

$$\underline{x}_2(k) = \underline{A}_{21}\underline{x}_1(k) + \underline{A}_{22}\underline{x}_2(k) + B_2\underline{u}(k) \qquad (38b)$$

$$\text{or } \underline{x}_2(k) = (I_f-\underline{A}_{22})^{-1}[A_{21}\underline{x}_1(k) + B_2\underline{u}(k)] \qquad (38c)$$

$$\text{where} \quad \begin{matrix} \underline{A}_{11} & \underline{A}_{12} \\ \underline{A}_{21} & \underline{A}_{22} \end{matrix} = \begin{matrix} A_{11} & hA_{12} \\ A_{21} & hA_{22} \end{matrix}$$

Rearranging (38), we get the slow subsystem as

$$\underline{x}_s(k+1) = A_o\underline{x}_s(k) + B_o\underline{u}_s(k) \qquad (39)$$

where, $\underline{x}_1(k) = \underline{x}_s(k)$, $\underline{x}_2(k)$ and $\underline{u}(k)$ are the slow components of the corresponding variables in (2), and

$$Ao = \underline{A}11 + \underline{A}12(I_f-\underline{A}_{22})^{-1}\underline{A}_{21}$$

$$B_o = B_1 + \underline{A}_{12}(I_f-\underline{A}_{22})^{-1}B_2$$

The fast subsystem is obtained by making the assumption that $\underline{x}_1(k) = x_s(k) =$ constant and $\underline{x}_2(k+1) = \underline{x}_2(k)$. From (2b) and (38c), we get the fast subsystem as,

$$x_f(k+1) = A_{21}x_f(k) + B_2u_f(k) \qquad (40)$$

where, $x_f(k) = x_2(k) - \underline{x}_2(k)$; $u_f(k) = u(k) - u_s(k)$;

3.5 Control of SPTS systems

The traditional control problems such as state feedback control design problem, eigenvalue assignment problem, observer design problem, are equally applicable to discrete-time systems with SPTS (Phillips 80, Mahmoud 82a,b,c, Mahmoud and Singh 81, 84, 85, Mahmoud et. el., 85, 86, Fernando and Nicholson 83a,b, Kando and Iwazumi 83a,b, 84, 85, Tran and Sawan 83a,b, 84a,b,c, Khorasani and Azim-Sadjadi 87). However, we will concentrate on the optimal control of these systems.

3.5.1 Open-Loop Optimal Control

Consider the slow-sampling model (2) having two-time scale character. The performance index to be minimized is

$$J = y(N)Sy(N) + 0.5 \sum_{0}^{N-1} [y(k)Qy(k) + u(k)Ru(k)] \qquad (41)$$

where $y(k) = [x_1(k), x_2(k)]$; S, and Q are real, positive semidefinite symmetric matrices of (n_1+n_2) dimensions, R is a real, positive definite matrix of order $r \times r$, N is a fixed integer indicating the terminal or final value of time.

Using the results of optimal control theory (Sage and White 77), the state $[x_1(k), x_2(k)]$ and costate $[p_1(k), p_2(k)]$ equations are obtained as

$$x_1(k+1) = A_{11}x_1(k) + hA_{12}x_2(k) - W_{11}p_1(k+1) - hW_{12}p_2(k+1)$$

$$x_2(k+1) = A_{21}x_1(k) + hA_{22}x_2(k) - W_{12}p_1(k+1) - hW_{22}p_2(k+1)$$

$$p_1(k) = Q_{11}x_1(k) + hQ_{12}x_2(k) + A_{11}p_1(k+1) + hA_{21}p_2(k+1)$$

$$p_2(k) = Q_{21}x_1(k) + hQ_{22}x_2(k) + A_{12}p_1(k+1) + hA_{22}p_2(k+1) \tag{42}$$

and the optimal control is given by

$$u(k) = -R^{-1}[B_1p_1(k+1) + hB_2p_2(k+1)] \tag{43}$$

where, $W_{ij} = B_iR^{-1}B_j$; $i,j = 1,2$

The $2(n_1+n_2)$ order two-point boundary value problem (TPBVP) represented by (42) which is in the singularly perturbed structure, is to be solved using the boundary conditions, $x_1(0)$, $x_2(0)$, $p_1(N)$ and $p_2(N)$.

The series representations for (41) are given by

$$x_1(k) = x_{1o}(k) + h^{k+1}x_{1i}(k) + h^{N-k+1}x_{1f}(k)$$

$$x_2(k) = x_{2o}(k) + h^{k}x_{2i}(k) + h^{N-k+1}x_{2f}(k) \tag{44}$$

$$p_1(k) = p_{1o}(k) + h^{k+1}p_{1i}(k) + h^{N-k+1}p_{1f}(k)$$

$$p_2(k) = p_{2o}(k) + h^{k+1}p_{2i}(k) + h^{N-k}p_{2f}(k)$$

where, $x_{1o}(k)$, $x_{2o}(k)$, $p_{1o}(k)$, and $p_{2o}(k)$ correspond to the outer solution, $x_{1i}(k)$, $x_{2i}(k)$, $p_{1i}(k)$, and $p_{2i}(k)$ correspond to the intial boundary layer correction, and $x_{1f}(k)$, $x_{2f}(k)$, $p_{1f}(k)$, and $p_{2f}(k)$ correspond to the final boundary layer correction. The details are found in Kando and Iwazumi (83b), Rajagopalan and Naidu (81), Naidu and Rao (85a), Rao and Naidu (82).

3.5.2 Closed-Loop Optimal Control

In this section, a two time scale discrete control system is considered. The closed-loop optimal linear quadratic regulator for the system requires solution of a full-order algebraic Riccati equation. Alternatively, the original system is decomposed into reduced-order slow and fast subsytems. The closed-loop optimal control of the subsystems requires the solution of two algebraic Riccati equations of order lower than that required for the full-order system. A composite, closed-loop suboptimal control is formed from the sum of the slow and fast feedback optimal controls. The main advantage of the method is a considerable reduction in the overall computational requirements for the closed-loop optimal control of digital systems (Naidu 77, Naidu and Rajagopalan 81, Rao and Naidu 82, Naidu and Rao 84, 85a, Othman et. et., 85, Kando 86, Naidu and Price 86).

4.2.1 Optimal Control of Original System

Consider the linear discrete system (2) having two-time scale character

The performance index to be minimized is

$$J = \underset{k=0}{} [y^T(k)Qy(k) + u^T(k)Ru(k)] \tag{45}$$

The closed-loop optimal control is given by (Sage and White 1977),

$$u(k) = -R^{-1}B^TP[I + BR^{-1}B^TP]^{-1}Ay(k) \tag{46}$$

where P, of order $(n_1+n_2)x(n_1+n_2)$, is the positive definite symmetric solution of matrix algebraic Riccati equation

$$P = A^TP[I + BR^{-1}B^TP]^{-1}A + Q \tag{47}$$

The closed-loop optimal system is given by

$$y(k+1) = (A - BF)y(k) \tag{48}$$

where, $F = R^{-1}B^TP[I + BR^{-1}B^TP]^{-1}A$

Instead of tackling the original regulator problem described by (2) and (45) directly, we decompose it appropriately into two regulator problems for slow and fast subsystems. For this, we first need to separate the original performance index into the sum of two performance indices for slow and fast subsytems.

The original performance index (45) has to be represented as the sum of the performance indices of the slow and fast subsytems. Using the transformation (6) between the original state variables, $[x_1(k)$ and $x_2(k)]$, and the subsystem variables, $[x_s(k)$ and $x_f(k)]$, in (45), and using $u_s(k) = u_f(k) = u(k)$, we get

$$J = [x_s^T(k)Q_{ss}x_s(k) + x_f^T(k)Q_{ff}x_f(k) + u_s^T(k)R_su_s(k)$$

$$x_s^T(k)Q_{sf}x_f(k) + x_f^T(k)Q_{fs}x_s(k) + u_f^T(k)R_f u_f(k)] \quad (49)$$

where, Q_{ss}, Q_{sf}, Q_{fs}, and Q_{ff} are related to Q_{ij} via L and M.

Since J has to be represented as the sum of J_s and J_f, we need to neglect Q_{sf} and Q_{fs}. Then

$$J_s = [x_s^T(k)Q_{ss}x_s(k) + u_s^T(k)R_s u_s(k)] \quad (50a)$$

$$J_f = [x_f^T(k)Q_{ff}x_f(k) + u_f^T(k)R_f u_f(k)] \quad (50b)$$

As we have neglected Q_{sf} and Q_{fs}, it certainly introduces an error, in that J will not be equal to the sum of J_s and J_f. To affect this, we need to readjust Q_{ss} and Q_{ff} (Othman et. el., 85).

As this is simply a design or synthesis approach, we can first select the performance indices of the subsystems and then formulate the original performance index. Thus, if

$$\underline{J}_s = [x_s^T(k)\underline{Q}_{ss}x_s(k) + u_s^T(k)\underline{R}_s u_s(k)] \quad (51)$$

$$\underline{J}_f = [x_f^T(k)\underline{Q}_{ff}x_f(k) + u_f^T(k)\underline{R}_f u_f(k)] \quad (52)$$

Using the transformation (7) between the subsystem variables $x_s(k)$ and $x_f(k)$ and the original system variables $x_1(k)$ and $x_2(k)$, we get

$$J = \underline{J}_s + \underline{J}_f$$

$$= [y^T(k)\underline{Q}y(k) + u^T(k)\underline{R}u(k)] \quad (53)$$

where, $\underline{Q}$ is related to $\underline{Q}_{ss}$ and $\underline{Q}_{ff}$ via L and M.

Thus, in (51)-(53), we first select $\underline{Q}_{ss}$, $\underline{Q}_{ff}$, $\underline{R}_s$ and $\underline{R}_f$, and then using L, and M, we get $\underline{Q}$ and $\underline{R}$. Here, we are able to decouple J and J_s and J_f exactly without any approximation. But the original J is dependent on L and M, the decoupling matrices, which may not be of practical advantage.

4.2.2 Optimal Control of Subsystems:

Using the transformation (6), we decompose the original system into slow and fast subsystems as

$$x_s(k+1) = A_s x_s(k) + B_s u_s(k) \quad (54)$$

$$x_f(k+1) = A_f x_f(k) + B_f u_f(k) \quad (55)$$

We now try to optimize these slow and fast subsystems with respect to their corresponding performance indices (50a) and (50b) respectively. The slow regulator problem consists of the slow subsytem (55) and the performance index (50a). The fast regulator problem consists of the fast subsystem and the performance index (50b). For convenience, we write $Q_{ss} = Q_s$; and $Q_{ff} = Q_f$.

The optimal feedback control of the slow subsytem is given by

$$u_s(k) = -R_s^{-1}B_s^TP_s[I_s + B_sR_s^{-1}B_s^TP_s]^{-1}A_sx_s(k) \tag{56}$$

where P_s is a positive definite symmetric solution of a reduced order algebraic Riccati equation,

$$P_s = A_sP_s[I_s + B_sR_s^{-1}B_s^TP_s]^{-1}A_s + Q_s \tag{57}$$

Similarly, the optimal feedback control of the fast subsytem becomes

$$u_f(k) = -R_f^{-1}B_f^TP_f[I_f + B_fR_f^{-1}B_f^TP_f]^{-1}A_fx_f(k) \tag{58}$$

where P_f is a positive definite symmetric solution of the reduced order algebraic Riccati equation

$$P_f = A_fP_f[I_f + B_fR_f^{-1}B_f^TP_f]^{-1}A_f + Q_f \tag{59}$$

Rewriting the control laws (55) and (57) as

$$u_s(k) = -F_sx_s(k) \tag{60}$$

$$u_f(k) = -F_fx_f(k) \tag{61}$$

We note that the control laws (56) and (58) are optimal with respect to the slow and fast subsystems (54) and (55) only. But, it is computationally simpler to determine these controls laws than the optimal control law (44) of the original system Kando and Iwazumi 83).

4.2.3 Composite Control:

The composite control is formulated as the sum of the slow and fast feedback controls given by (56) and (58). That is

$$u_c(k) = u_s(k) + u_f(k)$$

$$= -[F_sx_s(k) + F_fx_f(k)] \tag{62}$$

Using the transformation (7) between the slow and fast variables and the original variables in (62), we get

$$u_c(k) = -[F_{sc}x(k) + F_{fc}x(k)] = -F_cy(k) \tag{63}$$

where, F_{sc} and F_{fc} are related with F_c via L and M.

Using the composite control (63) in the original system,

$$y_c(k+1) = (A-BF_c)y_c(k) \tag{64}$$

It is known that minimizing the original performance index (44) with respect to the composite system (64) results in the suboptimal performance index (Othman et. el., 1985),

$$J_c = 0.5y^T(0)P_cy(0) \tag{65}$$

where P_c is the positive definite symmetric solution of discrete Lyapunov equation

$$P_c = (A-BF_c)^TP_c(A-BF_c) + Q + F_c^TRF_c \tag{66}$$

In an entirely different approach to the closed-loop optimal control of discrete systems possessing two-time scale character, the Riccati coefficient matrix P(k) is partitioned into singularly perturbed structure and the analysis is carried out on the Riccati equation (Naidu 77, Naidu and Rajagopalan 81, Kimura 83, Litkouhi 83, Litkouhi and Khalil 84, 85, Naidu and Rao 84, 85a, Kando and Iwazumi 83b, Naidu and Price 86, Kando 86). The theory of SPTS in adaptive systems and the optimal control of stochastic systems is considered by Delebeque and Quadrat (81), Ioannou and Kokotovic 82, Rao and Naidu (84)

5. Multirate Regulator Problem:

Singularly perturbed systems exhibit slow and fast behaviors. From an intuitive point of view, the measurement and control of the slow variables can be done at lower sampling rates in comparison with the fast variables (Litkouhi 83, Litkouhi and Khalil 84, 85, Kando 86, Kando and Iwazumi 86).

Consider the singularly perturbed continuous system (5) and the performance index (45). By the process of decomposition and discretization, the continuous system (5) is transformed to fast sampling model (10) or the slow sampling model (9) depending upon the discretizing interval. Similarly, the performance index (45) can be transformed. Using the slow (fast) sampling model (9) ((10)), and the corresponding performance index, we arrive at the slow (fast) sampling regulator problems, which are solved independently.

The slow sampling regulator problem is solved by decomposing it into slow and fast subproblems, where the fast subproblem exhibits a dead-beat behavior. Similarly, the fast sampling regulator problem is decomposed into slow and fast subproblems, where the slow subproblem, of continuous-time nature, dominates the system behavior over the whole interval.

In the singularly perturbed continuous system, $x_1(t,h)$ and $x_2(t,h)$ possess slow and fast behaviors respectively. Thus, the slow sampling rate (T_s) can be tolerated for measurement of the slow variable $x_1(t,h)$, i.e., $x_1(t,h)$ can be measured at slow rate $t = kT_s$ ($k=1,2,..$). On the other hand, $x_2(t,h)$ is measured at the fast rate $t = nT_f = k[1/h]T_f$.

By combining the controls of the slow subproblem of the slow sampling regulator and the fast subproblem of the fast sampling regulator, the multirate control is expressed as

$$u^c(nT_f,h) = u_s^s(k) + u_f^f(n)$$

$$= G_1x_1(kT_s,h) + G_2x_1(nT_f,h) + G_3x_2(nT_f,h) \quad (67)$$

Here, the states $x_1(kT_s,h)$ and $x_2(nT_f,h)$ are measurable. But, since the state $x_1(nT_f,h)$ can't be measured between $k[1/h]T_f < nT_f < (k+1)[1/h]T_f$, the above state feedback control can't be implemented. This difficulty is overcome by using the estimates of $x_1(nT_f,h)$. Finally, the multirate control is obtained as (Litkouhi and Khalil 85, Kando 86),

$$u^c(nT_f,h) = G_4x_1(kT_s,h) + G_5x_2(nT_f,h) \quad (67)$$

Figure shows the basic ideas behind the multirate control.

5. Conclusions:

In this paper we tried to overview the recent developments in the theory of singular perturbations and time scales (SPTS) in discrete control systems. The focus has been in three directions of modeling, analysis and control. In modeling, we reviewed sources of singualarly perturbed difference equations in their equivalent state space representations. Depending on the discretizing interval, we arrive at slow-sampling model and fast-sampling model. The analysis of two-time scale systems brought out the characteristic features of order reduction, boundary layer phenomena, stetching transformations, and correction series. In controlling the two-time scale systems, we addressed open-loop and closed-loop optimal control problems, highlighting the important issue of multirate sampling.

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14P.

IMPACT OF ATMOSPHERIC DENSITY SCALE HEIGHT ON THE PERFORMANCE OF AEROASSISTED COPLANAR ORBITAL TRANSFER VEHICLES

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NOMENCLATURE

AOTV	:	aeroassisted orbital transfer vehicle
C_D	:	drag coefficient
C_L	:	lift coefficient
g	:	acceleration of gravity
GEO	:	geosynchronous Earth orbit
h	=	$(r-r_s)/r_s$
h_s	:	reference altitude
$\underline{h}$	:	altitude
H	:	constant scale height
HEO	:	high Earth orbit
m	:	vehicle mass
OTV	:	orbital transfer vehicle
r	:	radial distance from Earth's center
r_s	:	reference radius
r_1	:	radius of HEO
r_2	:	radius of LEO
R	:	radius of spherical atmosphere
S	:	effective vehicle surface area
t	:	time
V	:	vehicle speed
γ	:	flight path angle
ΔV	:	impulsive change in V
μ	:	gravitational constant of Earth
ρ	:	atmospheric density
$\rho(r_s)$	:	value of ρ at r_s
$\rho(h_s)$	:	value of ρ at h_s

1. INTRODUCTION

The Space Transportation System (STS) is presently used for delivering payloads to Low Earth Orbit (LEO). Several of these payloads are transferred to High Earth Orbit (HEO), by expendable upper stage rockets that use either solid or liquid propellants. In order to support the deployment of a large number of satellites in Geosynchronous Earth Orbit (GEO) in an economical manner and to ultimately provide manned service, a reusable orbital transportation system is required. The Orbital Transfer Vehicle (OTV) is intended to transfer payloads from LEO to GEO and to return to LEO.

Since the concept of aeromaneuvering was first introduced about two decades ago[1], numerous studies have shown that performance advantages in terms of larger payloads, reduction in expenditure of energy and reusability, can be achieved using aerodynamic forces generated through atmospheric pass to get necessary orbital changes (both apogee and inclination) on the return leg, as compared with all propulsive orbital changes[2]. The concept of Aeroassisted Orbital Transfer Vehicle (AOTV), opens new mission opportunities, especially with regard to the initiation of a permanent space station.

Further, in a recent report of the National Commission on Space, PIONEERING THE SPACE FRONTIER, the concept of aerobraking for orbital transfer has been recognized as one of the seven critical technologies and recommended for demonstration projects in building the necessary technology base for pioneering the space frontier[3]. Broadly speaking, the two kinds of orbital transfer are coplanar orbital transfer[4] and orbital transfer with plane change[5].

2. COPLANAR ORBITAL TRANSFER

The coplanar transfer is from HEO to LEO using atmosphere to decrease the energy and thereby decrease the velocity of the vehicle[4]. Here, lift modulation is the only means of controlling the flight path in the atmosphere, propulsion being used only outside the atmosphere. The application of the thrust produces impulsive velocity changes (ΔV's) which are an indication of the fuel consumption for the orbital transfer.

The basic principle of coplanar orbital transfer from HEO to LEO is shown in Fig. 1. The in-plane tangential retroburn (ΔV_1) at HEO injects the vehicle into an elliptical orbit entering the atmosphere at point E. As the vehicle flies through the atmosphere, some of the kinetic energy is converted to heat, and consequently upon leaving the atmosphere at point F, the apogee of the orbit is decreased to the distance r_2. Finally at the new apogee, a second in-plane tangential burn (ΔV_2) is executed to circularize and thereby achieve the desired LEO. The minimum-fuel aeroassisted transfer is thus proportional to the minimum characteristic velocity, $\Delta V_1 + \Delta V_2$.

The basic equations of motion can be formulated in a variety of ways, depending on the independent variable. Using the time as the independent variable, we have[6]

$$\frac{dr}{dt} = V \sin v \tag{1a}$$

$$\frac{dV}{dt} = -\rho S_1 V^2 - (\mu/r^2)\sin v \tag{1b}$$

$$\frac{d\gamma}{dt} = \rho S_2 V - (\mu/r^2 V - V/r)\cos v \tag{1c}$$

where $S_1 = SC_D/2m$; $S_2 = SC_L/2m$

3. IMPACT OF SCALE HEIGHT

To analyze the effects of aerodynamic forces acting on a vehicle in flight, it is necessary to model the planetary atmosphere in which the flight takes place. The important feature of the atmosphere affecting the performance of the vehicle is the density. Hence, the main concern in modeling the atmosphere will be to conveniently and accurately represent the density.

A common way of representing the atmosphere is by a differential form[7]

$$d\rho/\rho = (-1/H)dr = (-1/H)d\underline{h} \tag{2}$$

where H is called the scale height. Using this differential form for the density, the atmosphere is characterized as a locally exponential function. If the coefficient H, is considered a constant over some small interval of altitude (or radius), the integration of Eq. (2) yields

$$\rho = \rho(r_s)\exp(-(r-r_s)/H) \quad (3a)$$

$$= \rho(h_s)\exp(-(h-h_s)/H) \quad (3b)$$

In the earlier works[4,6], 1962 US Standard Atmosphere[8] has been used, assuming a constant scale height over the entire interval of altitude of interest for AOTVs, ranging from 50 km to 120 km. Strictly speaking, the scale height is not constant over the entire interval, but changes depending upon the altitude.

In the present case, our approach has two features:

(i) As our interest of altitude is above 50 km, we try to use 1976 US Standard Atmosphere[9] which is the same as 1962 Standard Atmosphere below 50 km, but replaces the 1962 Standard Atmosphere at the higher altitudes.

(ii) In the exponential atmospheric model of Eq. (3), scale height has been assumed constant locally over a small interval of altitude. In other words, the scale height has been readjusted depending upon the interval $(r-r_s)$ used, instead of using a constant scale height over the entire interval of altitude ranging from 50 km to 120 km. This is believed to be more accurate to justify the integration of

Eq. (2) and hence the exponential atmospheric model of Eq. (3).

Using these two features, simulations are carried out for a coplanar orbital transfer vehicle. Firstly, we use the 1962 US Standard Atmosphere with the following parameter values[6]:

$\mu = 3.986\text{x}10^{14}\ m^2/sec^2$;	$m/S = 82\ kg/m^2$
$r_s = 6443$ km;	$C_L = 0.45$
$\rho(r_s) = 1.1\text{x}10^{-5}$;	$C_D = 1.54$
$H = 4.8$ km;	$r_e = 6378$ km

Secondly, simulations are carried out with adjustable scale height, instead of a constant scale height of 4.8 km. These simulations, carried out for a constant lift-drag ratio, are represented in a series of plots shown in Fig. 2-6, with constant scale height (—— line) and adjustable scale height (+ + + line). One would easily notice the appreciable difference between the two plots for altitude, velocity, flight path angle, density, heating effect, and so on. For example, in the altitude plot, a maximum discrepancy of 5800 meters occurs at about 110 seconds at an altitude of 70 km giving an 8-percent error.

Attempts are being made to carry out these simulations using (i) different formulations with altitude and energy as independent variables and (ii) shuttle-derived atmospheric data[10,11].

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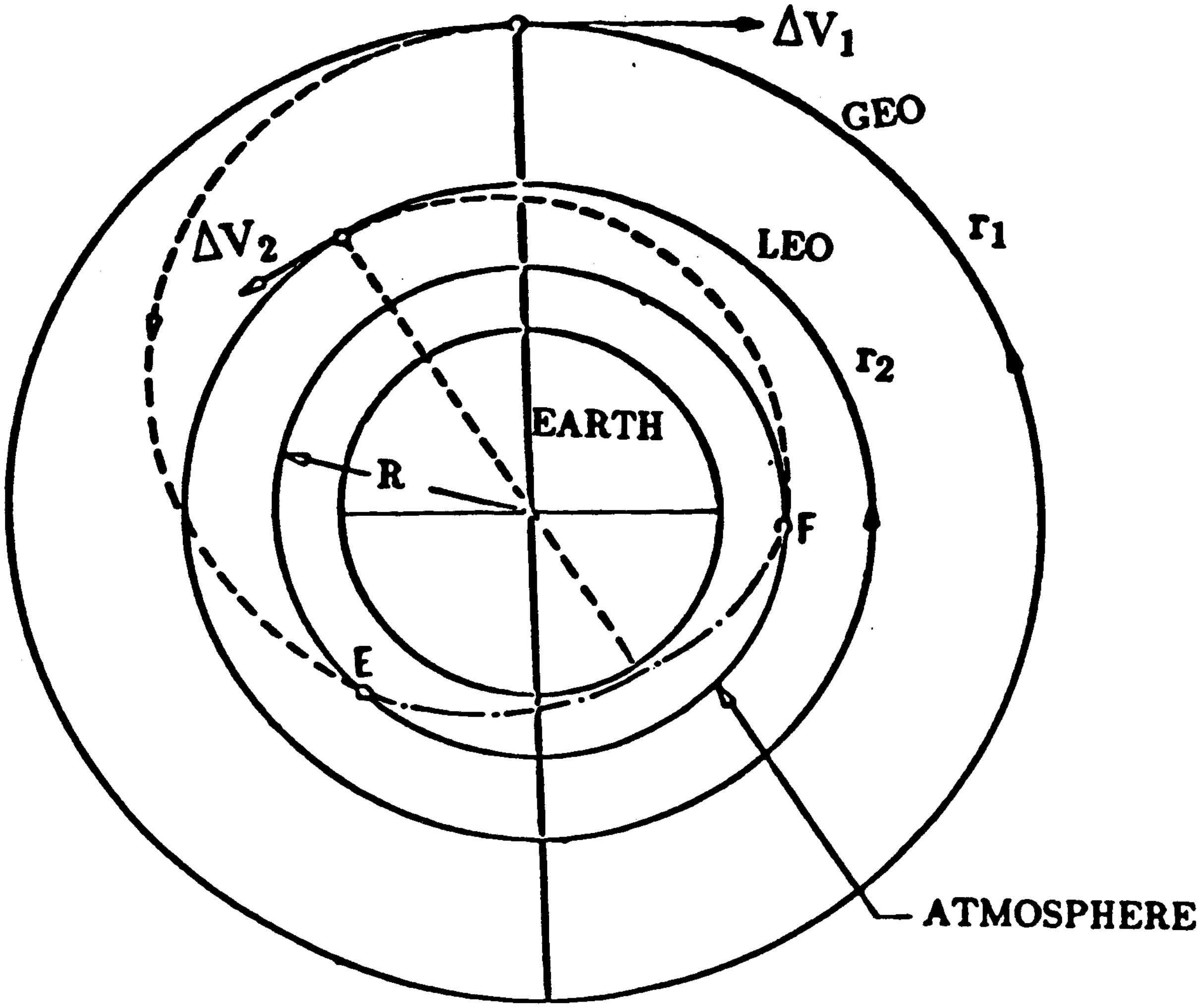

Fig. 1 Aeroassisted Coplanar Orbital Transfer

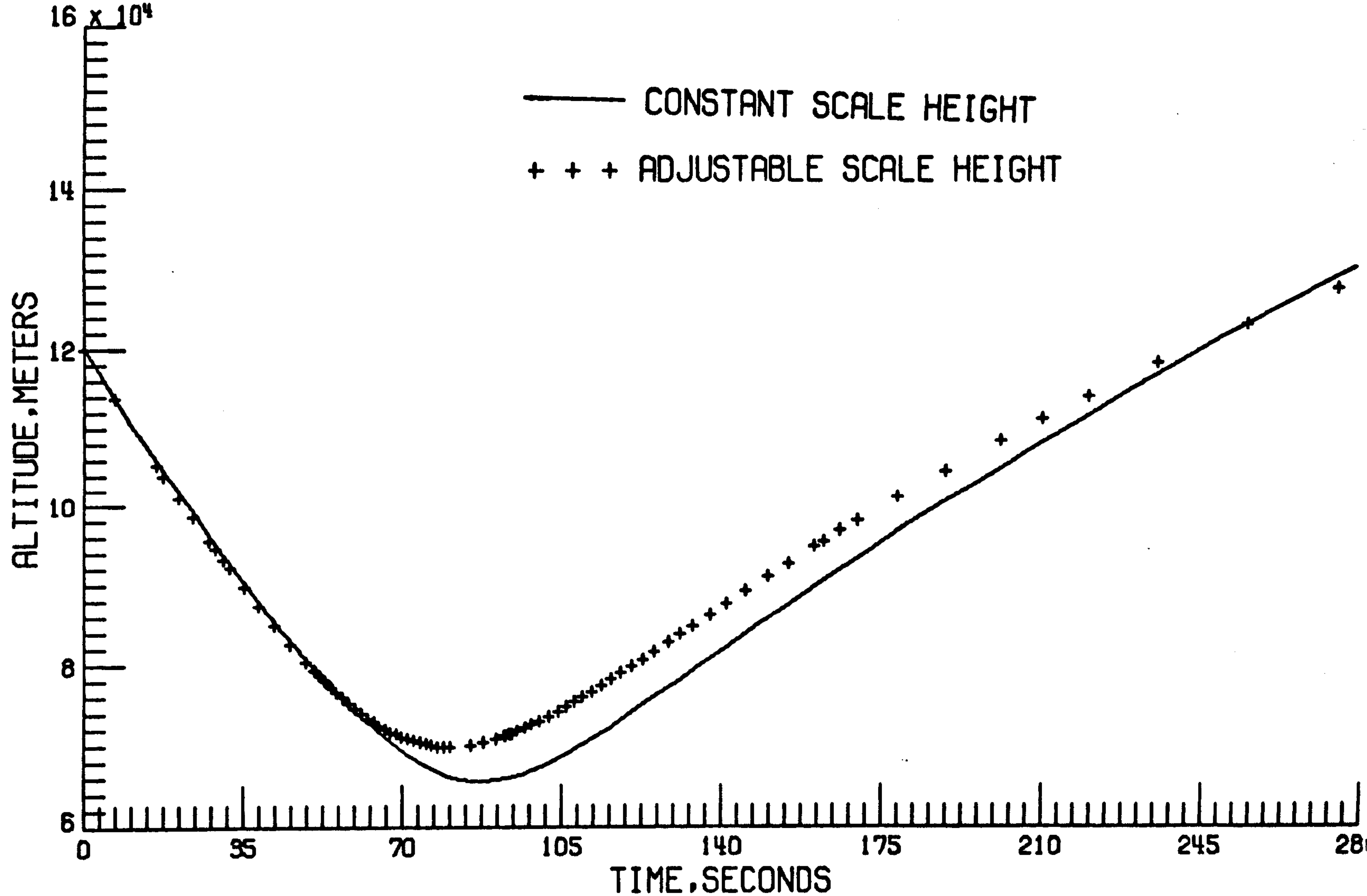

Fig. 2 Effect of Scale Height on Altitude

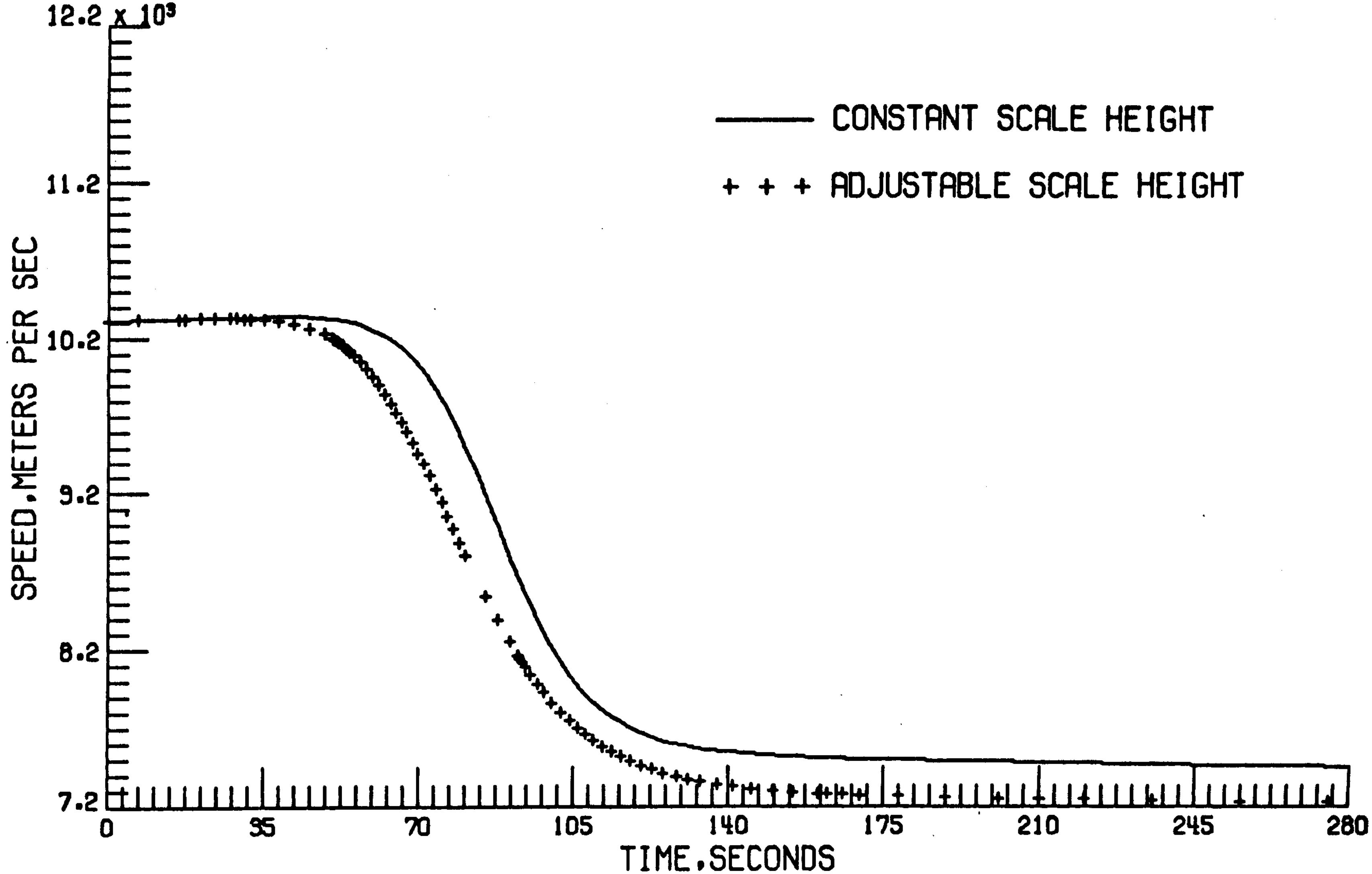

Fig. 3 Effect of Scale Height on Speed

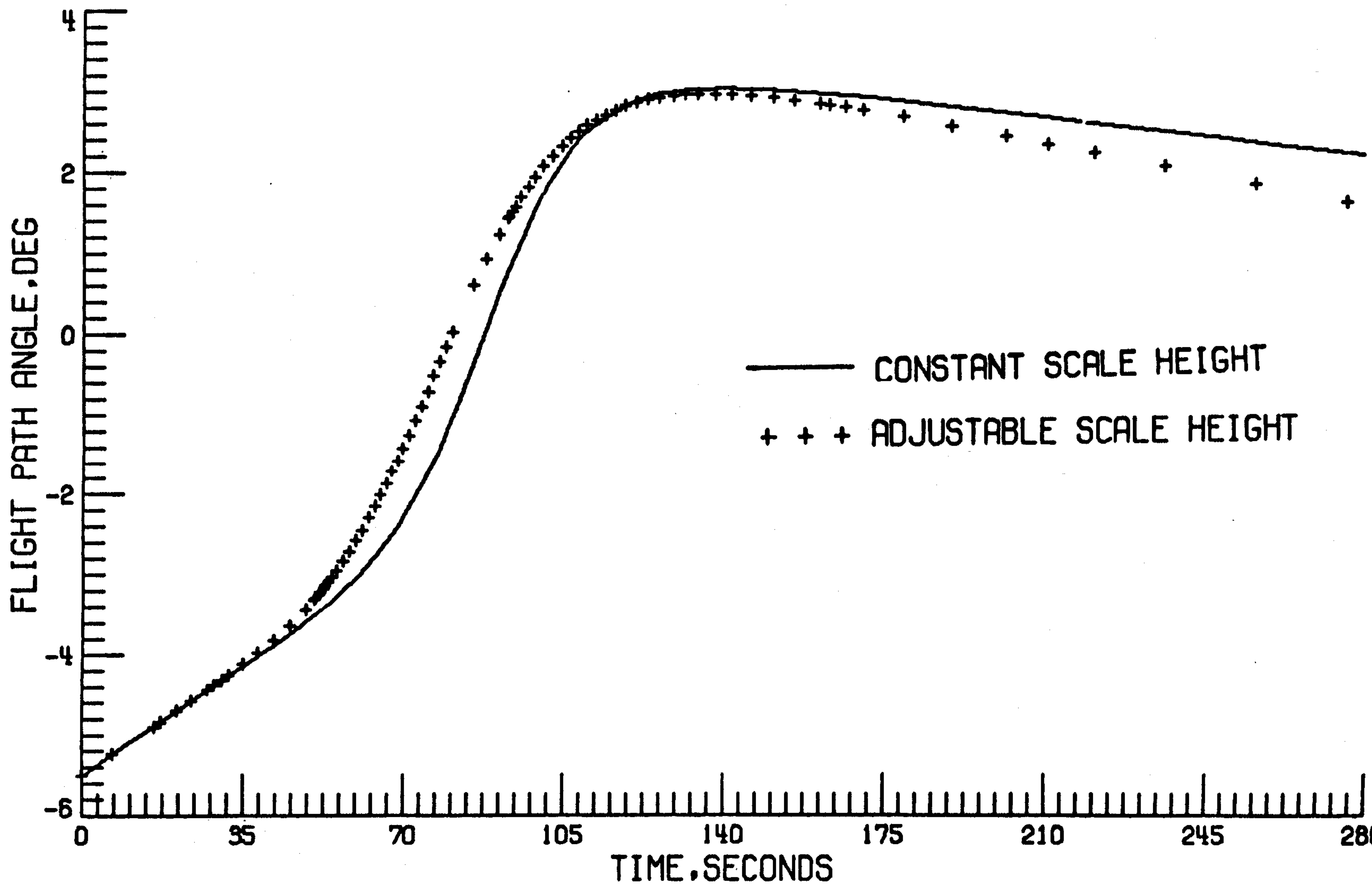

Fig. 4 Effect of Scale Height on Flight Path Angle

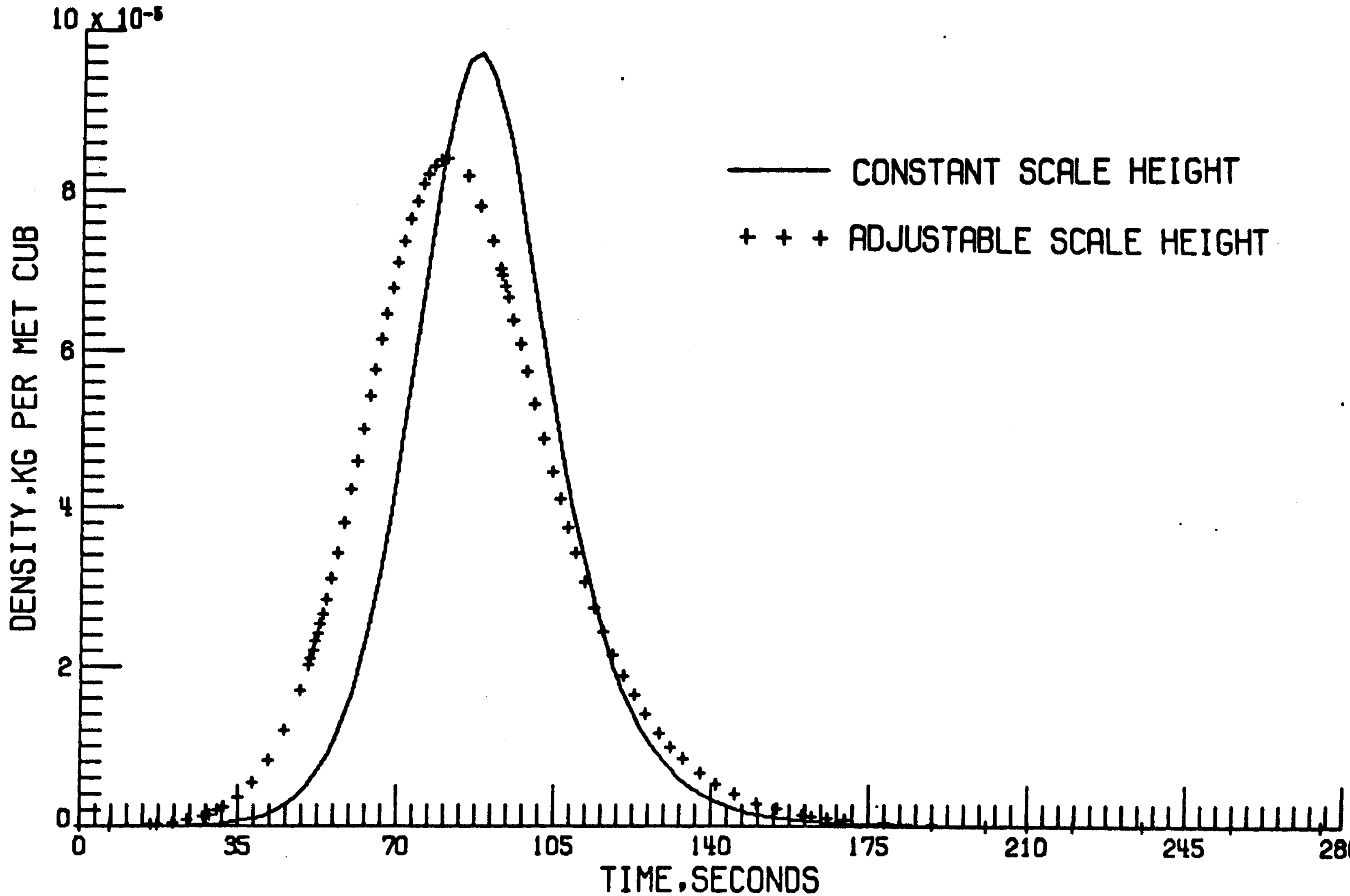

Fig. 5 Effect of Scale Height on Density

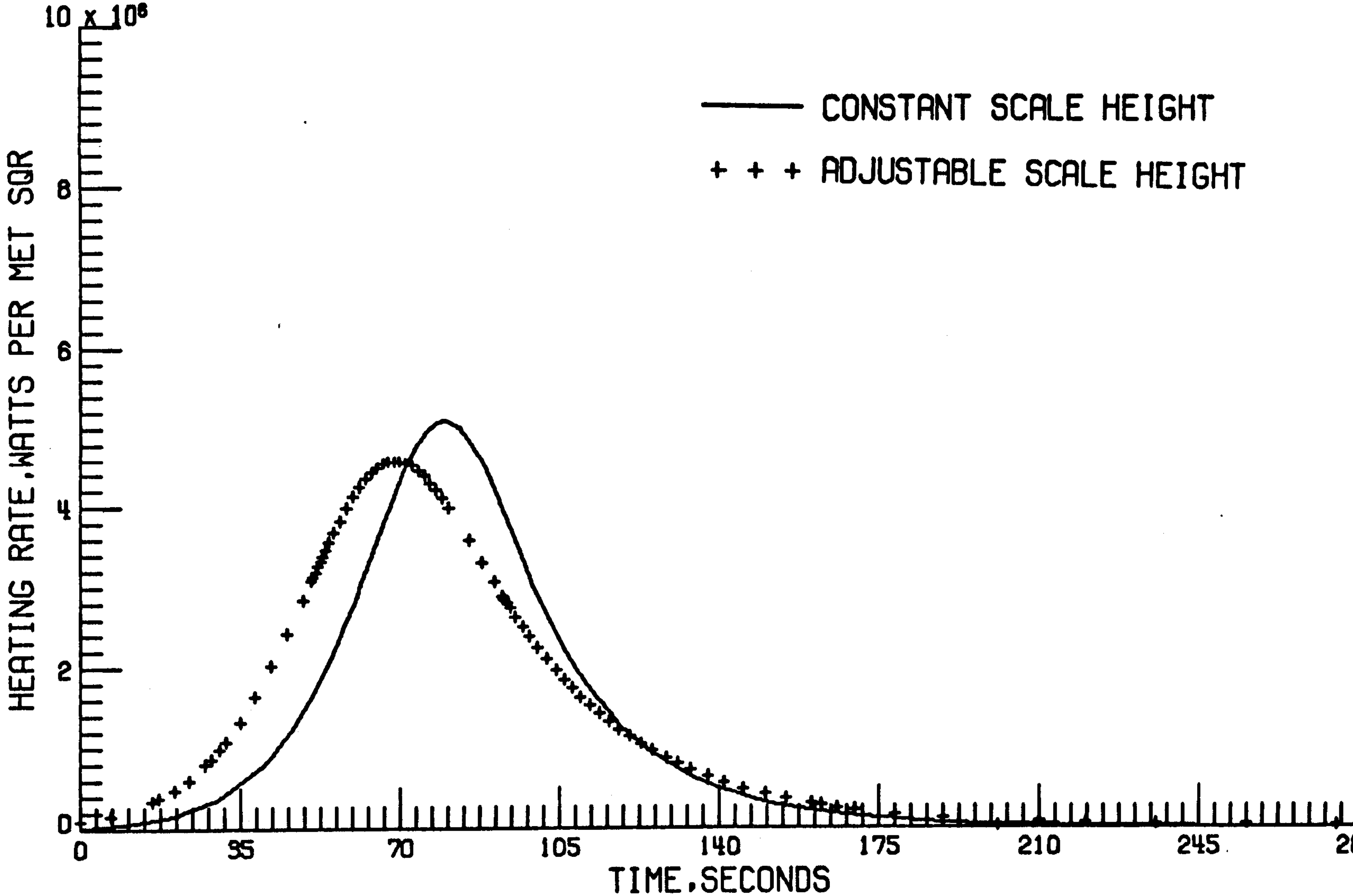

Fig. 6 Effect of Scale Height on Heating Rate

52-64

ON THE METHOD OF MATCHED ASYMPTOTIC EXPANSIONS

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Abstract: In the method of matched asymptotic expansions, a composite solution is constructed in terms of an outer solution an inner solution and a common solution. A critical examination of the method reveals that for a class of problems, the various terms of the common solution are formulated as polynomials in stretched variable without actually evaluating them from the outer solution. Incidentally, this also shows that the common solution of the method of matched asymptotic expansions is the same as the intermediate solution of singular perturbation method and that these two methods give identical results. Two illustrative examples are provided.

1. Introduction

Singular perturbation problems, where suppression of a small parameter affects order of the problems, have been solved by a wide variety of techniques [1-5]. Two of these techniques, singular perturbation method (SPM) [1,5] and the method of matched asymptotic expansions (MAE) [2,3] have been independently developed to a reasonable level of satisfaction. Essentially, the SPM consists of expressing total solution in terms of an outer solution an inner solution and an intermediate solution. On the other hand, in the method of MAE, a composite solution is constructed as the outer solution, the inner solution and a common solution. These techniques have been so far thought to be somewhat independent and their advantages and disadvantages have been discussed in their applications to fluid mechanics and flight mechanics [6-9].

In this paper, a critical examination of the method of matched asymptotic expansions reveals that the various terms of the common solution of MAE can be generated as polynomials in stretched variable without actually solving for them from the outer solution as it is done presently. This also shows that the common solution of the method of MAE and the intermediate solution of the SPM are the same and hence that these methods give identical results for a certain class of problems. Two illustrative examples are given.

2. Method of Matched Asymptotic Expansions

The method of matched asymptotic expansions has been extensively used in fluid mechanics [2]. In this method, a composite solution is expressed as an outer solution, plus an inner solution, and minus a common solution. The outer solution is valid outside the boundary layer and the inner solution is valid inside the boundary layer. Then both inner and outer solutions are common over an overlap region. The common solution is obtained by using a matching principle, which is stated in a variety of ways. The matching principle also enables us to evaluate the undetermined constants of outer and inner solutions.

We describe briefly the method of MAE as applicable to initial value problems. Consider

$$\frac{dx}{dt} = f(x,\ z,\ \varepsilon,\ t) \tag{1a}$$

$$\varepsilon\frac{dz}{dt} = g(x,\ z,\ \varepsilon,\ t) \tag{1b}$$

where x, and z are n-and m-dimensional state vectors respectively and ε is a small positive parameter responsible for singular perturbation. We begin by representing the solutions in the form of a series in powers of ε as

$$x(t,\varepsilon) = \sum_{i=0}^{\infty} x^{(i)}(t)\varepsilon^{i}; \qquad z(t,\varepsilon) = \sum_{i=0}^{\infty} z^{(i)}(t)\varepsilon^{i} \tag{2}$$

and determine the various terms $x^{(i)}(t)$ and $z^{(i)}(t)$ by means of formal substitution of (2) in (1) and comparison of coefficients of equal powers of ε. Then the following set of recursive equations are obtained. For zeroth order approximation,

$$\frac{dx^{(0)}(t)}{dt} = f^{0}\ [x^{(0)}(t),\ z^{(0)}(t),\ 0,\ t] \tag{3a}$$

$$0 = g^{0}\ [x^{(0)}(t),\ z^{(0)}(t),\ 0,\ t] \tag{3b}$$

and for first order approximation, we have

$$\frac{dx^{(1)}(t)}{dt} = f^{1}\ [x^{(1)}(t),\ z^{(1)}(t),\ x^{(0)}(t),\ z^{(0)}(t),\ t] \tag{4a}$$

$$\frac{dz^{(0)}(t)}{dt} = g^{1}\ [x^{(1)}(t),\ z^{(1)}(t),\ x^{(0)}(t),\ z^{(0)}(t),\ t] \tag{4b}$$

where the notation f^{0}, and f^{1} is used to indicate all the terms on the right hand side. Note that the zeroth order problem (3) is the same as the degenerate problem obtained by making $\varepsilon = 0$ in (1), and a boundary layer is said to exist at $t = 0$. Since the series (2) corresponds to the solution outside the boundary layer, it is called an outer series.

The solution of (3) is obtained by using $x^{(0)}(t=0) = x(0)$; and in general $z^{(0)}(t=0) \neq z(0)$. On the otherhand, the solution of (4) poses a problem, since the initial condition $x^{(1)}(t=0)$ is not yet known. Once $x^{(1)}(t)$ is solved for, $z^{(1)}(t)$ is automatically known from (4b). In order to relate the outer series (2) to the solution of (1) in the boundary layer, we use a stretching transformation

$$\tau = t/\varepsilon \tag{5}$$

Then using (5) in (1), the stretched or inner problem becomes

$$\frac{d\bar{x}(\tau)}{d\tau} = f[\bar{x}(\tau),\ \bar{z}(\tau),\ \varepsilon,\ \varepsilon\tau] \tag{6a}$$

$$\frac{d\bar{z}(\tau)}{d\tau} = g[\bar{x}(\tau),\ \bar{z}(\tau),\ \varepsilon,\ \varepsilon\tau] \tag{6b}$$

This has inner series expansions of the form

$$\bar{x}(\tau,\varepsilon) = \sum_{i=0}^{\infty} \bar{x}^{(i)}(\tau)\varepsilon^{i}; \qquad \bar{z}(\tau,\varepsilon) = \sum_{i=0}^{\infty} \bar{z}^{(i)}(\tau)\varepsilon^{i} \tag{7}$$

Substitution of (7) in (6) and comparison of coefficients result in for zeroth order approximation as

$$\frac{d\bar{x}^{(0)}(\tau)}{d\tau} = 0 \tag{8a}$$

$$\frac{d\bar{z}^{(0)}(\tau)}{d\tau} = \bar{g}^{0}[\bar{x}^{(0)}(\tau),\ \bar{z}^{(0)}(\tau)] \tag{8b}$$

and for first order approximation

$$\frac{d\bar{x}^{(1)}(\tau)}{d\tau} = \bar{f}^{0}[\bar{x}^{(0)}(\tau),\ \bar{z}^{(0)}(\tau)] \tag{9a}$$

$$\frac{d\bar{z}^{(1)}(\tau)}{d\tau} = \bar{g}^{1}[\bar{x}^{(1)}(\tau),\ \bar{z}^{(1)}(\tau),\ \bar{x}^{(0)}(\tau),\ \bar{z}^{(0)}(\tau)] \tag{9b}$$

The inner problem (6)-(9) has initial conditions as

$$\bar{x}^{(0)}(\tau=0) = x(t=0); \quad \bar{z}^{(0)}(t=0) = z(t=0) \tag{10a}$$

$$\bar{x}^{(i)}(\tau=0) = 0; \quad \bar{z}^{(i)}(\tau=0) = 0; \quad i > 0 \tag{10b}$$

Still, we have not resolved the problem of determining the initial value $x^{(1)}(t=0)$ of the outer equation (4). This is done by using a matching principle of the method of MAE. Thus the matching principle is stated as

inner expansion of outer solution =
outer expansion of inner solution (11)

To include higher approximations, we have

the j^{th}-term inner expansion of the k^{th}-term outer solution =
the k^{th}-term outer expansion of the j^{th}-term inner solution (12)

where j and k are any two integers. In practice, j is usually chosen as either k or k+1. Here, inner expansion of outer solution $(x^{o})^{i}$ is obtained by extending the outer solution so that it approaches the boundary layer. This is done by first transforming the independent variable t to that of the inner variable $\tau = t/\varepsilon$ and then expanding it in powers of ε.. Similarly, the inner solution is extended so that it approaches beyond the boundary layer. This is done by first transforming the inner variable τ to that of the outer variable $t = \varepsilon\tau$. The solution is then expanded in powers of ε. This results in $(x^i)^o$, the outer expansion of inner solution. A suitable choice of undetermined coefficients will be given by the matching principle

$$(x^o)^i = (x^i)^o \tag{13}$$

To any order approximation, the composite solution x_c is given by

$$\begin{aligned} x_c &= x^o + x^i - (x^o)^i \\ &= x^o + x^i - (x^i)^o \end{aligned} \tag{14}$$

where x^o, and x^i are the outer and inner solutions respectively to any order of approximation and $(x^o)^i = (x^i)^o$ is also called the common solution. Similar expressions can be given for z also.

3. An Examination of Common Solution

In this section, we will show that the common solution defined as the inner expansion of the outer solution is simply formulated as a polynomial in the stretched variable. The steps involved in obtaining the common solution are (i) express the outer solution in the inner variable τ, (ii) expand it around ε □ 0, and (iii) rearrange the resulting solution in powers of ε. Thus, consider the outer solution as

$$x^o(t) = x^{(0)}(t) + \varepsilon x^{(1)}(t) + \ldots\ldots \tag{15}$$

We express this outer solution in the inner variable $\tau = t/\varepsilon$ as

$$x^o(\varepsilon\tau) = x^{(0)}(\varepsilon\tau) + \varepsilon x^{(1)}(\varepsilon\tau) + \ldots\ldots \tag{16}$$

Expanding (21) around $\varepsilon = 0$, we get

$$(x^{\circ})^{i} = \left[x^{(0)}(\varepsilon\tau)\Big|_{\varepsilon=0} + \varepsilon\frac{\partial x^{(0)}(\varepsilon\tau)}{\partial\varepsilon}\Big|_{\varepsilon=0} + \ldots\right] +$$

$$\varepsilon\left[x^{(1)}(\varepsilon\tau)\Big|_{\varepsilon=0} + \varepsilon\frac{\partial x^{(1)}(\varepsilon\tau)}{\partial\varepsilon}\Big|_{\varepsilon=0} + \ldots\right] \tag{17}$$

Now evaluation of function $x^{(i)}(\varepsilon\tau)$ at $\varepsilon = 0$ in τ- plane is the same as its evaluation at t = 0 in t-plane, and the partial derivative of function $x^{(i)}(\varepsilon\tau)$, with respect to ε in τ-plane is the same as its partial derivative w.r.t. t multiplied by τ in t-plane. Thus,

$$(x^{\circ})^{i} = \left[x^{(0)}(t=0) + \varepsilon\tau\frac{\partial x^{(0)}(t)}{\partial t}\Big|_{t=0} + \ldots\right] +$$

$$\varepsilon\left[x^{(1)}(t=0) + \varepsilon\tau\frac{\partial x^{(1)}(t)}{\partial t}\Big|_{t=0} + \ldots\right]$$

$$= x^{(0)}(t=0) + \varepsilon\left[x^{(1)}(t=0) + \tau\frac{\partial x^{(0)}(t)}{\partial t}\Big|_{t=0}\right] + \ldots$$

$$= x^{(0)}(0) + \varepsilon\left[x^{(1)}(t=0) + \tau\dot{x}^{(0)}(0)\right] + \ldots\ldots$$

$$(x^{\circ})^{i} = \tilde{x}^{(0)}(\tau) + \varepsilon\tilde{x}^{(1)}(\tau) + \ldots\ldots \tag{18}$$

where,

$$\left.\begin{aligned} \tilde{x}^{(0)}(\tau) &= x^{(0)}(0) \\ \tilde{x}^{(1)}(\tau) &= x^{(1)}(0) + \tau\dot{x}^{(0)}(0) \end{aligned}\right\} \tag{19}$$

Here, the dot over x denotes differentiation of x w. r. t. t . Similar expression can be obtained for the function z. Let us note that the intermediate solution of SPM is obtained by (i) expanding the outer solution around $t = 0$, (ii) expressing it in the inner variable τ, and (iii) rearranging the resulting solution in powers of ε [1,5]. Then, the common solution (18) of the method of MAE is found to be the same as the intermediate solution of the SPM. Thus, the outer and inner solutions being the same in the SPM and the method of MAE, we clearly see that these two methods give identical results. Essentially, this equivalence means that the expansion of the outer solution around $t = 0$ and transformation into τ-plane is the same as transformation of the outer solution into τ-plane first and then expansion around $\varepsilon = 0$. The main advantage of the present formulation of the common solution is that its various terms can be very easily generated as polynomials in τ and hence one need not have explicit outer solution to arrive at the common solution.

In this way, we suggest an improved method of MAE, where the outer and inner solutions are obtained as before and the common solution is generated simply as a polynomial in the stretched variable τ, instead of evaluating it from the explicit outer solutions as it is done usually [2].

4. Examples

We give two examples, one on an initial value problem, and the other on a boundary value problem [10].

<u>Example 1: Initial Value Problem</u>

Consider a simple second order system so that we can get explicit expressions for the solutions.

$$\frac{dx}{dt} = z \qquad x(t=0) = a \tag{20a}$$

$$\varepsilon\frac{dz}{dt} = -x - z \qquad z(t=0) = b \tag{20b}$$

Applying the method of MAE described in Section 2, we summarize the results as follows. The outer solutions corresponding to (3) and (4) are

$$\left.\begin{aligned} x^{(0)}(t) &= ae^{-t}; \quad z^{(0)}(t) = -ae^{-t} \\ x^{(1)}(t) &= [x^{(1)}(0) - at]e^{-t} \\ z^{(1)}(t) &= [-x^{(1)}(0) + at - a]e^{-t} \end{aligned}\right\} \tag{21}$$

The inner solutions corresponding to (8) and (9) are

$$\left.\begin{aligned} \bar{x}^{(0)}(\tau) &= a; \quad \bar{z}^{(0)}(\tau) = -a + (a + b)e^{-\tau} \\ \bar{x}^{(1)}(\tau) &= (a + b) - a\tau - (a + b)e^{-\tau} \\ \bar{z}^{(1)}(\tau) &= -(2a + b) + a\tau + [2a + b + (a + b)\tau]e^{-\tau} \end{aligned}\right\} \tag{22}$$

Considering the two-term expansions only, the common solution (CS) for x is obtained as

$$(CS) = (x^{i})^{o} = (x^{o})^{i} \tag{23}$$

From (22), we obtain $(x^{i})^{o}$, the outer expansion of the inner solution by first expressing the inner solution in the outer variable $t = \varepsilon\tau$ and then expanding it around $\varepsilon = 0$. Thus

$$x^{i}(\tau) = a + \varepsilon[(a + b) - a\tau - (a + b)e^{-\tau}]$$

$$(x^{i})^{o} = a(1 - t) + \varepsilon(a + b) \tag{24}$$

Next, from (21), we obtain $(x^{o})^{i}$, the inner expansion of the outer solution as

$$x^{o}(t) = ae^{-t} + \varepsilon[x^{(1)}(0) - at]e^{-t}$$

$$(x^{o})^{i} = a(1 - t) + \varepsilon x^{(1)}(0) \tag{25}$$

Alternatively, in the improved approach, we formulate $(x^{o})^{i}$ as

$$(x^{o})^{i} = x^{(0)}(t=0) + \varepsilon[x^{(1)}(t=0) + \tau\dot{x}^{(0)}(t=0)]$$

$$= a + \varepsilon[x^{(1)}(0) + \tau(-a)]$$

$$= a(1 - t) + \varepsilon x^{(1)}(0) \qquad (26)$$

Equating (24) and (25), we get the value of undetermined coefficient $x^{(1)}(0)$ as

$$x^{(1)}(0) = (a + b) \qquad (27)$$

Similarly for z, we have

$$(CS) = (z^{i})^{o} = (z^{i})^{o} \qquad (28)$$

From (22), we obtain $(z^{i})^{o}$, the outer expansion of the inner solution as

$$z^{i}(\tau) = [-a + (a + b)e^{-\tau}] +$$

$$\varepsilon[-(2a + b) + a\tau + \{2a + b + (a + b)\tau\}e^{-\tau}]$$

$$(z^{i})^{o} = -a(1 - t) + \varepsilon[-(2a + b)] \qquad (29)$$

Next, we obtain $(z^{o})^{i}$, the inner expansion of the outer solution as

$$z^{o}(t) = -ae^{-t} + \varepsilon[z^{(1)}(0) + at]e^{-t}$$

$$(z^{o})^{i} = -a(1 - t) + \varepsilon z^{(1)}(0) \qquad (30)$$

Alternatively, in the improved method, we formulate $(z^{o})^{i}$ as

$$(z^{o})^{i} = z^{(0)}(t=0) + \varepsilon[z^{(1)}(t=0) + \tau\dot{z}^{(0)}(t=0)]$$

$$= -a + \varepsilon[z^{(1)}(0) + \tau a]$$

$$= -a(1 - t) + \varepsilon z^{(1)}(0) \tag{31}$$

Using (28)-(30), we get the value of the undetermined coefficient $z^{(1)}(0)$ as

$$z^{(1)}(0) = -(2a + b) \tag{32}$$

The composite solution corresponding to (14) is

$$x_c(t,\varepsilon) = ae^{-t} + \varepsilon[(a + b)(e^{-t} - e^{-t/\varepsilon}) - ate^{-t}] \tag{33a}$$

$$z_c(t,\varepsilon) = -ae^{-t} + (a + b)(1 + t)e^{-t/\varepsilon} +$$

$$\varepsilon[(2a + b)(e^{-t/\varepsilon} - e^{-t}) + ate^{-t}] \tag{33b}$$

<u>Example 2. Boundary Value Problem</u>

We present a boundary value problem and obtain all solutions upto second order (two-term) approximation [10].

Consider

$$\frac{dx}{dt} = z \qquad x(t=0) = a; \quad x(t=1) = b \tag{34}$$

$$\varepsilon\frac{dz}{dt} = - x - z$$

Since the boundary conditions are imposed on x only, it is enough if we summarize the solutions for x only. The outer solutions are

$$\left.\begin{aligned} x^{(0)}(t) &= be^{(1-t)} \\ x^{(1)}(t) &= [x^{(1)}(0) - bet]e^{-t} \end{aligned}\right\} \tag{35}$$

The inner solution are

$$\left.\begin{aligned}\bar{x}^{(0)}(\tau) &= be + (a - be)e^{-\tau} \\ \bar{x}^{(1)}(\tau) &= [be - be\tau + \{(a - be)\tau - be\}e^{-\tau}]\end{aligned}\right\} . \tag{36}$$

The common solution, obtained as outer expansion of inner solution, is

$$(x^{i})^{o} = be(1 - t) + \varepsilon be \tag{37}$$

Similarly, the common solution, obtained as the inner expansion of the outer solution, is

$$(x^{o})^{i} = be(1 - t) + \varepsilon x^{(1)}(0) \tag{38}$$

Alternatively, in the modified method, we formulate $(x^{o})^{i}$ as

$$\begin{aligned}(x^{o})^{i} &= x^{(0)}(t=0) + \varepsilon[x^{(1)}(t=0) + \tau\dot{x}^{(0)}(t=0)] \\ &= be + \varepsilon[x^{(1)}(0) - be\tau] \\ &= be(1 - t) + \varepsilon x^{(1)}(0)\end{aligned} \tag{39}$$

Let us note that the common solution (38) for $(x^{o})^{i}$ can be easily generated as shown by (39). Equating (37) with (38) or (39), we get the value of undetermined coefficient $x^{(1)}(0) = be$. Finally, the composite solution is given by

$$\begin{aligned}x_{c}(t, \varepsilon) = [1 + \varepsilon(1 - t)]be^{(1-t)} &+ \\ [(a - be)(1 + t) - \varepsilon be]e^{-t/\varepsilon}&\end{aligned} \tag{40}$$

5. Conclusions

In this paper, a critical examination of the method of MAE have revealed that the terms of the common solution could be generated as polynomials in stretched variable without actually solving for them as it is done presently. We have also seen that the common solution of the method of MAE is the same as the intermediate solution of the SPM and hence these two methods give identical results. Two examples have been given for illustration.

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